II YEARS - B.SC., MATHEMATICS

M. S. University
Directorate of Distance &
Continuing Education
Tirunelveli

FOURIER SERIES AND INTEGRAL TRANSFORMS

Prepared by: Dr. R. Thayalarajan

மனோன்மணியம் சுந்தரனார் பல்கலைக்கழகம்

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தொலைநிலை தொடர் கல்வி இயக்ககம்

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FOURIER TRANSFORMS AND INTEGRAL TRANSFORMS

SYLLABUS

Unit I:

Fourier Series – Definitions – Fourier Coefficients and Fourier Series for a given periodic function with 2π and $2l$, odd and even functions – Convergence of Fourier Series.

Unit II:

Half range Fourier series – Parseval's theorem – Root – Mean Square value of a function – Harmonic analysis – Complex form of Fourier series.

Unit III:

Fourier Transforms – Fourier Integral Theorem – Fourier Sine and Cosine Transforms – Properties of Fourier Transform – Convolution Theorem – Parsevel's Identity.

Unit IV:

Laplace Transforms – Definition – Results – Laplace Transform of Periodic Function – Some general Theorems – Evaluation of certain Integrals.

Unit V:

The Inverse Transform – Results – Solving Ordinary differential equations with constant coefficients, simultaneous linear differential equations and differential equations with variable coefficients by Laplace Transform.

Text Book:

1. T. Veerarajan, Engineering Mathematics, Tata Mcgraw Hill, New Delhi 2001.
2. T.K. Manikkavasagam Pillai and S. Narayanan, Differential Equations and its Applications, S. Viswanathan Printers pvt limited, 2002.

UNIT – I

FOURIER SERIES

Introduction:

Fourier series named after the French Mathematician cum Physicist Jean Baptiste Joseph Fourier (1768 – 1830), has several interesting applications in engineering problems. He introduced Fourier series in 1822 while he was investigating the problem of heat conduction. This series became a very important tool in Mathematics. In this chapter we discuss the basic concepts relating to Fourier Series and obtain development of several functions.

Periodic Function:

A function $f(x)$ is said to have a period T if for all x, $f(x+T) = f(x)$, Where T is a positive constant. The least value of $T > 0$ is called the period of $f(x)$.

For Example:

The Trigonometry functions are periodic functions.

- $\sin x$ then the function has periods $2\pi, 4\pi, 6\pi, \ldots$
- $\cos x$ then the function has period $2\pi, 4\pi, 6\pi, \ldots$
- $\tan x$ then the function has period π

1. Show that a constant has any positive number as period.

Solution:

$f(x) = k$

$f(x+c) = k$

$f(x) = f(x+c)$

Therefore, $f(x)$ is periodic with period c.

Limit of a Function:

A function $f(x)$ is said to tend to a limit l as $x \to a$ if to each given $\varepsilon > 0$, there exists a positive number δ such that $|f(x) - l| < \varepsilon$, when $0 < |x - a| < \delta$. It is denoted by $\lim_{x \to a} f(x) = l$.

Continuous Function:

A function $f(x)$ is said to be continuous at $x = a$ if $f(a-0) = f(a+0) = f(a)$. (ie) $f(x)$ is said to be continuous in an interval (a, b) if it is continuous at every point of the interval.

Discontinuous Function:

A function $f(x)$is said to be discontinuous at a point if it is not continuous at that point.

Ex. $f(x) = \begin{cases} x & if\ x < 1 \\ x^2 & if\ \ x > 1 \end{cases}$. Here $x = 1$ is a point of discontinuity.

PIECEWISE CONTINUOUS FUNCTION:

Definition 1:

A function $f(x)$ is said to be piecewise continuous in an interval if

- the interval can be divided into a finite number of subintervals in each of which $f(x)$ is continuous and
- the limits of $f(x)$ as x approaches the end points of each subinterval are finite.

Definition 2:

A piecewise continuous function is one that has atmost a finite number of finite discontinuities.

Dirichlet Conditions

A function $f(x)$ is defined in $c < x < c + 2l$ can be expanded as an infinite trigonometric series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

provided by

(i) $f(x)$ is defined and single valued except possibly at a finite number of points in $(c, c + 2l)$.

(ii) $f(x)$ is periodic in $(c, c + 2l)$.

(iii) $f(x)$ and $f'(x)$ are piecewise continuous in $(c, c + 2l)$.

(iv) $f(x)$ has no or finite number of maxima or minima in $(c, c + 2l)$.

FOURIER SERIES:

The infinite trigonometry series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

is called the Fourier series of $f(x)$ which satisfy Dirichlet conditions in $c \le x \le c + 2l$.

where a_0, a_n and b_n are called **Fourier coefficients** and the values are given by Euler's formula. Then we have

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)dx$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}dx$$

2. Write the Euler's formula of $f(x)$ in $(c, c+2\pi)$.

$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\;dx$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\;dx$$

Problem based in the interval $(0, 2l)$

3. Find the Fourier Series expansion of period $2l$ for the function $f(x) = (l-x)^2$ in the range $(0, 2l)$. Deduce the sum of the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

Solution:

Given: $f(x) = (l-x)^2$ in $(0,2l)$

w.k.t. the Fourier Series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{--------------------- (1)}$$

To find a_0:

w.k.t. $a_0 = \frac{1}{l}\int_0^{2l} f(x)dx$

$$= \frac{1}{l}\int_0^{2l}(l-x)^2\,dx$$

$$= \frac{1}{l}\left[\frac{(l-x)^3}{-3}\right]_0^{2l}$$

$$= \frac{1}{l}\left[-\frac{(l-2l)^3}{3} + \frac{(l-0)^3}{3}\right]$$

$$= \frac{1}{l}\left[\frac{l^3}{3} + \frac{l^3}{3}\right]$$

$$= \frac{1}{l}\left[\frac{2l^3}{3}\right]$$

$$a_0 = \frac{2l^2}{3} \quad \text{---------------------- (2)}$$

To find a_n:

w.k.t. $a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}dx$

$$= \frac{1}{l}\int_0^{2l}(l-x)^2\cos\frac{n\pi x}{l}dx$$

By using Bernoulli's Formula $\int uv dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$

Take $u = (l-x)^2$ $\qquad v = \cos\frac{n\pi x}{l}$

$u' = 2(l-x)(-1)$ $\qquad v_1 = \frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}$

$= -2(l-x)$ $\qquad v_2 = -\frac{\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}$

$u'' = -2(-1)$ $\qquad v_3 = -\frac{\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}$

$= 2$

$$= \frac{1}{l}\left[(l-x)^2\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) + 2(l-x)\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + 2\left(-\frac{\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^{2l}$$

$$= \frac{1}{l}\left\{\left[(l-2l)^2\left(\frac{\sin\frac{n\pi(2l)}{l}}{\frac{n\pi}{l}}\right) - 2(l-2l)\left(\frac{\cos\frac{n\pi(2l)}{l}}{\frac{n^2\pi^2}{l^2}}\right) - 2\left(\frac{\sin\frac{n\pi(2l)}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]\right.$$
$$\left. - \left[(l-0)^2\left(\frac{\sin\frac{n\pi(0)}{l}}{\frac{n\pi}{l}}\right) - 2(l-0)\left(\frac{\cos\frac{n\pi(0)}{l}}{\frac{n^2\pi^2}{l^2}}\right) - 2\left(\frac{\sin\frac{n\pi(0)}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]\right\}$$

$$= \frac{1}{l}\left\{\left[(-l)^2\left(\frac{\sin 2n\pi}{\frac{n\pi}{l}}\right) + 2l\left(\frac{\cos 2n\pi}{\frac{n^2\pi^2}{l^2}}\right) - 2\left(\frac{\sin 2n\pi}{\frac{n^3\pi^3}{l^3}}\right)\right] - \left[-2l\left(\frac{\cos 0}{\frac{n^2\pi^2}{l^2}}\right)\right]\right\}$$

$$= \frac{1}{l}\left\{\left[(-l)^2\left(\frac{l(0)}{n\pi}\right) + 2l\left(\frac{l^2(-1)^2}{n^2\pi^2}\right) - 2\left(\frac{l^3(0)}{n^3\pi^3}\right)\right] - \left[-2l\left(\frac{l^2}{n^2\pi^2}\right)\right]\right\}$$

$\because \sin n\pi = 0$ and $\cos n\pi = (-1)^n$

$$= \frac{1}{l}\left\{\left[2l\left(\frac{l^2}{n^2\pi^2}\right)\right] - \left[-2l\left(\frac{l^2}{n^2\pi^2}\right)\right]\right\}$$

$$= \frac{1}{l}\left\{\frac{2l^3}{n^2\pi^2} + \frac{2l^3}{n^2\pi^2}\right\}$$

$$= \frac{1}{l}\left\{\frac{4l^3}{n^2\pi^2}\right\}$$

$$\boldsymbol{a_n = \frac{4l^2}{n^2\pi^2}} \quad \textbf{----------------------- (3)}$$

To find b_n:

w.k.t. $b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$

$$= \frac{1}{l}\int_0^{2l} (l-x)^2 \sin\frac{n\pi x}{l}\,dx$$

By using Bernoulli's Formula $\int uvdx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$

Take $u = (l-x)^2$ $\qquad v = \sin\frac{n\pi x}{l}$

$u' = 2(l-x)(-1)$ $\qquad v_1 = -\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}$

$= -2(l-x)$ $\qquad v_2 = -\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}$

$u'' = -2(-1)$ $\qquad v_3 = \frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}$

$= 2$

$$= \frac{1}{l}\left[(l-x)^2\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 2(l-x)\left(-\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + 2\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^{2l}$$

$$= \frac{1}{l}\left\{\left[(l-2l)^2\left(-\frac{\cos\frac{n\pi(2l)}{l}}{\frac{n\pi}{l}}\right) - 2(l-2l)\left(-\frac{\sin\frac{n\pi(2l)}{l}}{\frac{n^2\pi^2}{l^2}}\right) + 2\left(\frac{\cos\frac{n\pi(2l)}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]\right.$$
$$\left. - \left[(l-0)^2\left(-\frac{\cos\frac{n\pi(0)}{l}}{\frac{n\pi}{l}}\right) - 2(l-0)\left(-\frac{\sin\frac{n\pi(0)}{l}}{\frac{n^2\pi^2}{l^2}}\right) + 2\left(\frac{\cos\frac{n\pi(0)}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]\right\}$$

$$= \frac{1}{l}\left\{\left[(-l)^2\left(-\frac{\cos 2n\pi}{\frac{n\pi}{l}}\right) + 2l\left(-\frac{\sin 2n\pi}{\frac{n^2\pi^2}{l^2}}\right) + 2\left(\frac{\cos 2n\pi}{\frac{n^3\pi^3}{l^3}}\right)\right]\right.$$
$$\left. - \left[(l)^2\left(-\frac{\cos 0}{\frac{n\pi}{l}}\right) - 2l\left(-\frac{\sin 0}{\frac{n^2\pi^2}{l^2}}\right) + 2\left(\frac{\cos 0}{\frac{n^3\pi^3}{l^3}}\right)\right]\right\}$$

$$= \frac{1}{l}\left\{\left[(-l)^2\left(-\frac{l(-1)^2}{n\pi}\right) + 2l\left(-\frac{l^2(0)}{n^2\pi^2}\right) + 2\left(\frac{l^3(-1)^2}{n^3\pi^3}\right)\right] - \left[(l)^2\left(-\frac{l}{n\pi}\right) - 2l\left(-\frac{l^2(0)}{n^2\pi^2}\right) + 2\left(\frac{l^3}{n^3\pi^3}\right)\right]\right\}$$

$\because \sin n\pi = 0$ and $\cos n\pi = (-1)^n$

$$= \frac{1}{l}\left\{\left[(-l)^2\left(-\frac{l(-1)^2}{n\pi}\right) + 2\left(\frac{l^3(-1)^2}{n^3\pi^3}\right)\right] - \left[(l)^2\left(-\frac{l}{n\pi}\right) + 2\left(\frac{l^3}{n^3\pi^3}\right)\right]\right\}$$

$\boldsymbol{b_n = 0}$ **---------------------- (4)**

Substituting (2), (3) and (4) in (1), we get

$$f(x) = \frac{l^2}{3} + \sum_{n=1}^{\infty} \frac{4l^2}{n^2\pi^2} \cos\frac{n\pi x}{l}$$

$$= \boldsymbol{\frac{l^2}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l}} \quad \textbf{------------------- (5)}$$

Here 0 is a point of discontunity which is an end point of the given interval $0 < x < 2l$. Therefore, the sum of Fourier series (5) at $x = 0$ is the average value of $f(x)$ at the end points. i.e., at $x = 0$ and at $x = 2l$.

Put $x = 0$ in (5) we get

$$\frac{l^2}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l} = \frac{f(0)+f(2l)}{2} \qquad \begin{bmatrix} \because f(x) = (l-x)^2 \\ f(0) = (l-0)^2 = l^2 \\ f(2l) = (l-2l)^2 = l^2 \end{bmatrix}$$

$$= \frac{l^2+l^2}{2}$$

$$= l^2$$

$$\frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = l^2 - \frac{l^2}{3}$$

$$\frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{2l^2}{3}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{2\pi^2}{12}$$

$$\boldsymbol{\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}} \qquad \textbf{(or)}$$

$$\boldsymbol{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \infty = \pi^2/6}$$

4. Find the Fourier series of period $2l$ for the function $f(x) = x(2l - x)$ in $(0, 2l)$. Hence deduce the sum of $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

Solution:

Given: $f(x) = x(2l - x)$ in $(0,2l)$

w.k.t. the Fourier Series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{-------------------- (1)}$$

To find a_0:

w.k.t. $a_0 = \frac{1}{l}\int_0^{2l} f(x)dx$

$$= \frac{1}{l}\int_0^{2l} x(2l - x)\,dx$$

$$= \frac{1}{l}\int_0^{2l}(2lx - x^2)\,dx$$

$$= \frac{1}{l}\left[\frac{2lx^2}{2} - \frac{x^3}{3}\right]_0^{2l}$$

$$= \frac{1}{l}\left[4l^3 - \frac{8l^3}{3}\right]$$

$$= \frac{1}{l}\left[\frac{12l^3 - 8l^3}{3}\right]$$

$$= \frac{1}{l}\left[\frac{4l^3}{3}\right]$$

$$a_0 = \frac{4l^2}{3} \quad \text{---------------------- (2)}$$

To find a_n:

w.k.t. $a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$

$$= \frac{1}{l}\int_0^{2l} x(2l - x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left\{2l\int_0^{2l} x\cos\frac{n\pi x}{l}\,dx - \int_0^{2l} x^2\cos\frac{n\pi x}{l}\,dx\right\}$$

By using Bernoulli's Formula $\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$

Take $u = x$ $\quad v = \cos\frac{n\pi x}{l}$ $\quad u = x^2$

$u' = 1$ $\quad v_1 = \frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}$ $\quad u' = 2x$

$u'' = 0$ $\quad v_2 = -\frac{\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}$ $\quad u'' = 2$

$v_3 = -\frac{\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}$ $\quad u''' = 0$

$$= \frac{1}{l}\left\{2l\left[x\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 1\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^{2l} - \left[x^2\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 2x\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + 2\left(-\frac{\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^{2l}\right\}$$

$$= \frac{1}{l}\left\{2l\left[2l\left(\frac{\sin\frac{n\pi(2l)}{l}}{\frac{n\pi}{l}}\right) + \left(\frac{\cos\frac{n\pi(2l)}{l}}{\frac{n^2\pi^2}{l^2}}\right) - 0\left(\frac{\sin\frac{n\pi(0)}{l}}{\frac{n\pi}{l}}\right) - \left(\frac{\cos n\pi(0)}{\frac{n^2\pi^2}{l^2}}\right)\right] - \left[4l^2\left(\frac{\sin\frac{n\pi(2l)}{l}}{\frac{n\pi}{l}}\right) + 4l\left(\frac{\cos\frac{n\pi(2l)}{l}}{\frac{n^2\pi^2}{l^2}}\right) - 2\left(\frac{\sin\frac{n\pi(2l)}{l}}{\frac{n^3\pi^3}{l^3}}\right) - (0)^2\left(\frac{\sin\frac{n\pi(0)}{l}}{\frac{n\pi}{l}}\right) + 2(0)\left(\frac{\cos\frac{n\pi(0)}{l}}{\frac{n^2\pi^2}{l^2}}\right) - 2\left(\frac{\sin\frac{n\pi(0)}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]\right\}$$

$$= \frac{1}{l}\left\{2l\left[2l\left(\frac{\sin 2n\pi}{\frac{n\pi}{l}}\right) + \left(\frac{\cos 2n\pi}{\frac{n^2\pi^2}{l^2}}\right) - 0\left(\frac{\sin 0}{\frac{n\pi}{l}}\right) - \left(\frac{\cos n\pi(0)}{\frac{n^2\pi^2}{l^2}}\right)\right] - \left[4l^2\left(\frac{\sin 2n\pi}{\frac{n\pi}{l}}\right) + 4l\left(\frac{\cos 2n\pi}{\frac{n^2\pi^2}{l^2}}\right) - 2\left(\frac{\sin 2n\pi}{\frac{n^3\pi^3}{l^3}}\right) - (0)^2\left(\frac{\sin 0}{\frac{n\pi}{l}}\right) + 2(0)\left(\frac{\cos 0}{\frac{n^2\pi^2}{l^2}}\right) - 2\left(\frac{\sin 0}{\frac{n^3\pi^3}{l^3}}\right)\right]\right\}$$

$$= \frac{1}{l}\left\{2l\left[2l\left(\frac{l(0)}{n\pi}\right) + \left(\frac{l^2(-1)^2}{n^2\pi^2}\right) - 0 - \left(\frac{l^2}{n^2\pi^2}\right)\right]\right.$$

$$\left. - \left[4l^2\left(\frac{l(0)}{n\pi}\right) + 4l\left(\frac{l^2(-1)^2}{n^2\pi^2}\right) - 2\left(\frac{l^3(0)}{n^3\pi^3}\right) - (0)^2\left(\frac{l(0)}{n\pi}\right) + 2(0)\left(\frac{l^2}{n^2\pi^2}\right) - 2\left(\frac{l^3(0)}{n^3\pi^3}\right)\right]\right\}$$

$\because \sin n\pi = 0$ and $\cos n\pi = (-1)^n$

$$= \frac{1}{l}\left\{2l\left[\left(\frac{l^2}{n^2\pi^2}\right) - \left(\frac{l^2}{n^2\pi^2}\right)\right] - \left[4l\left(\frac{l^2}{n^2\pi^2}\right)\right]\right\}$$

$$= \frac{1}{l}\left\{-\frac{4l^3}{n^2\pi^2}\right\}$$

$$\boldsymbol{a_n = -\frac{4l^2}{n^2\pi^2}} \text{ ------------------------ } \mathbf{(3)}$$

To find b_n:

w.k.t. $b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}dx$

$$= \frac{1}{l}\int_0^{2l} x(2l - x)\cos\frac{n\pi x}{l}dx$$

$$= \frac{1}{l}\left\{2l\int_0^{2l} x\cos\frac{n\pi x}{l}dx - \int_0^{2l} x^2\cos\frac{n\pi x}{l}dx\right\}$$

By using Bernoulli's Formula $\int uv dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$

Take $u = x$	$v = \sin\frac{n\pi x}{l}$	$u = x^2$
$u' = 1$	$v_1 = -\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}$	$u' = 2x$
$u'' = 0$	$v_2 = -\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}$	$u'' = 2$
	$v_3 = \frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}$	$u''' = 0$

$$= \frac{1}{l}\left\{2l\left[x\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 1\left(-\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^{2l}\right.$$

$$\left. - \left[x^2\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 2x\left(-\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + 2\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^{2l}\right\}$$

$$= \frac{1}{l}\left\{2l\left[2l\left(-\frac{\cos\frac{n\pi(2l)}{l}}{\frac{n\pi}{l}}\right) + \left(\frac{\sin\frac{n\pi(2l)}{l}}{\frac{n^2\pi^2}{l^2}}\right) - 0\left(-\frac{\cos\frac{n\pi(0)}{l}}{\frac{n\pi}{l}}\right) - \left(\frac{\sin n\pi(0)}{\frac{n^2\pi^2}{l^2}}\right)\right]\right.$$

$$- \left[4l^2\left(-\frac{\cos\frac{n\pi(2l)}{l}}{\frac{n\pi}{l}}\right) + 4l\left(-\frac{\sin\frac{n\pi(2l)}{l}}{\frac{n^2\pi^2}{l^2}}\right) + 2\left(\frac{\cos\frac{n\pi(2l)}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right.$$

$$\left.\left. - (0)^2\left(-\frac{\cos\frac{n\pi(0)}{l}}{\frac{n\pi}{l}}\right) + 2(0)\left(-\frac{\sin\frac{n\pi(0)}{l}}{\frac{n^2\pi^2}{l^2}}\right) - 2\left(\frac{\cos\frac{n\pi(0)}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]\right\}$$

$$= \frac{1}{l}\left\{2l\left[2l\left(-\frac{\cos 2n\pi}{\frac{n\pi}{l}}\right) + \left(\frac{\sin 2n\pi}{\frac{n^2\pi^2}{l^2}}\right) - 0\left(-\frac{\cos 0}{\frac{n\pi}{l}}\right) - \left(\frac{\sin 0}{\frac{n^2\pi^2}{l^2}}\right)\right]\right.$$

$$- \left[4l^2\left(-\frac{\cos 2n\pi}{\frac{n\pi}{l}}\right) + 4l\left(-\frac{\sin 2n\pi}{\frac{n^2\pi^2}{l^2}}\right) + 2\left(\frac{\cos 2n\pi}{\frac{n^3\pi^3}{l^3}}\right) - (0)^2\left(-\frac{\cos 0}{\frac{n\pi}{l}}\right)\right.$$

$$\left.\left. + 2(0)\left(-\frac{\sin 0}{\frac{n^2\pi^2}{l^2}}\right) - 2\left(\frac{\cos 0}{\frac{n^3\pi^3}{l^3}}\right)\right]\right\}$$

$$= \frac{1}{l}\left\{2l\left[2l\left(-\frac{l}{n\pi}\right) + \left(\frac{l^2(0)}{n^2\pi^2}\right) - 0\left(-\frac{l}{n\pi}\right) - \left(\frac{l^2(0)}{n^2\pi^2}\right)\right]\right.$$

$$\left. - \left[4l^2\left(-\frac{l}{n\pi}\right) + 4l\left(-\frac{l^2(0)}{n^2\pi^2}\right) + 2\left(\frac{l^3}{n^3\pi^3}\right) - (0)^2\left(-\frac{l}{n\pi}\right) + 2(0)\left(-\frac{l^2(0)}{n^2\pi^2}\right) - 2\left(\frac{l^3}{n^3\pi^3}\right)\right]\right\}$$

$\because \sin n\pi = 0$ and $\cos n\pi = (-1)^n$

$$= \frac{1}{l}\left\{2l\left[-\frac{2l^2}{n\pi}\right] - \left[-\frac{4l^3}{n\pi} + \frac{2l^3}{n^3\pi^3} - \frac{2l^3}{n^3\pi^3}\right]\right\}$$

$$= \frac{1}{l}\left\{-\frac{4l^3}{n^2\pi^2} + \frac{4l^3}{n^2\pi^2}\right\}$$

$\boldsymbol{b_n = 0}$ **----------------------- (4)**

Substituting (2), (3) and (4) in (1), we get

$$f(x) = \frac{\frac{4l^2}{3}}{2} + \sum_{n=1}^{\infty} -\frac{4l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}$$

$$= \boldsymbol{\frac{2l^2}{3} - \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l}} \text{ ------------------- } \mathbf{(5)}$$

Put $x = l$ lies in $(0, 2l)$ and is a point of continuity of the function $f(x) = x(2l - x)$.

$$\frac{2l^2}{3} - \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l} = x(2l - x)$$

$$\frac{2l^2}{3} - \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi = l(2l - l)$$

$$-\frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi = l^2 - \frac{2l^2}{3}$$

$$-\frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi = \frac{l^2}{3}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi = \frac{l^2}{3} \times -\frac{\pi^2}{4l^2}$$

$$\frac{1}{1^2}\cos\pi + \frac{1}{2^2}\cos 2\pi + \frac{1}{3^2}\cos 3\pi + \frac{1}{4^2}\cos 4\pi + \cdots = -\frac{\pi^2}{12}$$

$$\frac{1}{1^2}(-1) + \frac{1}{2^2}(-1)^2 + \frac{1}{3^2}(-1)^3 + \frac{1}{4^2}(-1)^4 + \cdots = -\frac{\pi^2}{12}$$

$$-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots = -\frac{\pi^2}{12}$$

$$-\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right) = -\frac{\pi^2}{12}$$

$$\boldsymbol{\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}}$$

5. **Find the Fourier series of the function** $f(x) = \begin{cases} 0 & if\ -\pi < x < 0 \\ \sin x & if\ 0 \le x < \pi \end{cases}$**. Hence deduce that (i)** $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots + \infty$

(ii) $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \cdots + \infty$

Solution:

Given: $f(x) = \begin{cases} 0 & if\ -\pi < x < 0 \\ \sin x & if\ 0 \le x < \pi \end{cases}$ in $(-\pi, \pi)$

w.k.t. the Fourier Series expansion is

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ -------------------- **(1)**

To find a_0:

w.k.t. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$

$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\, dx + \int_{0}^{\pi} f(x)\, dx\right\}$

$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} 0\, dx + \int_{0}^{\pi} \sin x\, dx\right\}$

$= \frac{1}{\pi}[-\cos x]_0^{\pi}$

$= \frac{1}{\pi}[-\cos \pi + \cos 0]$

$a_0 = \frac{2}{\pi}$ --------------------- **(2)**

To find a_n:

w.k.t. $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$

$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x) \cos nx\, dx + \int_{0}^{\pi} f(x) \cos nx\, dx\right\}$

$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} (0) \cos nx\, dx + \int_{0}^{\pi} \sin x \cos nx\, dx\right\}$

$= \frac{1}{\pi}\left\{\int_{0}^{\pi} \sin x \cos nx\, dx\right\}$

$= \frac{1}{\pi}\left\{\int_{0}^{\pi} \cos nx \sin x\, dx\right\}$

$$= \frac{1}{\pi}\left\{\int_0^\pi \left[\frac{sin(n+1)x - sin(n-1)x}{2}\right] dx\right\}$$

$$= \frac{1}{2\pi}\left\{\int_0^\pi sin(n+1)x\ \ dx - \int_0^\pi sin(n-1)x\ \ dx\right\}$$

$$= \frac{1}{2\pi}\left\{\left[-\frac{cos(n+1)x}{n+1}\right]_0^\pi - \left[-\frac{cos(n-1)x}{n-1}\right]_0^\pi\right\}$$

$$= \frac{1}{2\pi}\left\{-\frac{cos(n+1)\pi}{n+1} + \frac{cos(n+1)(0)}{n+1} + \frac{cos(n-1)\pi}{n-1} - \frac{cos(n-1)(0)}{n-1}\right\}$$

$$= \frac{1}{2\pi}\left\{-\frac{(-1)^{n+1}}{n+1} + \frac{1}{n+1} + \frac{(-1)^{n-1}}{n-1} - \frac{1}{n-1}\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{-(-1)^{n+1}(n-1)+(n-1)+(-1)^{(n-1)(n+1)}-(n+1)}{(n+1)(n-1)}\right\}$$

$$= \frac{1}{2\pi(n^2-1)}\{(-1)^n(n-1) - (-1)^n(n+1) - 2\}$$

$$= \frac{1}{2\pi(n^2-1)}\{n(-1)^n - (-1)^n - n(-1)^n - (-1)^n - 2\}$$

$$= \frac{1}{2\pi(n^2-1)}\{-2(-1)^n - 2\}$$

$$= \frac{1}{\pi(n^2-1)}\{-(-1)^n - 1\} \qquad \because (-1)^{n+1} = (-1)^{n-1}$$

$$= \frac{1}{\pi(n^2-1)}\{(-1)^{n-1} - 1\}$$

$$\boldsymbol{a_n} = \begin{cases} \mathbf{0} & \boldsymbol{if\ n\ is\ odd} \\ -\dfrac{\mathbf{2}}{\boldsymbol{\pi(n^2-1)}} & \boldsymbol{if\ n\ is\ even} \end{cases} \text{ ----------------------- } \mathbf{(3)}$$

To find b_n:

w.k.t. $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, sin\, nx\, dx$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\, sin\, nx\, dx + \int_0^\pi f(x)\, sin\, nx\, dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} (0)\, sin\, nx\, dx + \int_0^\pi sin\, x\, sin\, nx\, dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^\pi sin\, x\, sin\, nx\, dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^\pi \left[\frac{cos(1+n)x - cos(1-n)x}{2}\right] dx\right\}$$

$$= \frac{1}{2\pi}\left\{\int_0^\pi cos(1+n)x\ \ dx - \int_0^\pi cos(1-n)x\ \ dx\right\}$$

$$= \frac{1}{2\pi}\left\{\left[\frac{sin(1+n)x}{1+n}\right]_0^\pi - \left[\frac{sin(1-n)x}{1-n}\right]_0^\pi\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{sin(1+n)\pi}{1+n} + \frac{sin(1+n)(0)}{1+n} - \frac{sin(1-n)\pi}{1-n} - \frac{sin(1-n)(0)}{1-n}\right\}$$

$$= \frac{1}{2\pi}\{0\}$$

$\boldsymbol{b_n = 0}$ **----------------------- (4)**

Put $n = 1$ in (2), we get

$$(2) \Rightarrow a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos x\, dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\, cos\, x\, dx + \int_0^\pi f(x)\, cos\, x\, dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} (0)\, cos\, x\, dx + \int_0^\pi sin\, x\, cos\, x\, dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^\pi sin\, x\, cos\, x\, dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^\pi \frac{sin\, 2x}{2} dx\right\}$$

$$= \frac{1}{2\pi}\left[-\frac{cos\, 2x}{2}\right]_0^\pi$$

$$= \frac{1}{4\pi}[-\, cos\, 2\pi + cos\, 0]$$

$$= \frac{1}{4\pi}[-1 + 1]$$

$\boldsymbol{a_1 = 0}$ **--------------------------- (5)**

Put $n = 1$ in (3), we get

$$(3) \Rightarrow b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, sin\, x\, dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\, sin\, x\, dx + \int_0^\pi f(x)\, sin\, x\, dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0}(0)\, sin\, x\, dx + \int_{0}^{\pi} sin\, x\, sin\, x\, dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_{0}^{\pi} sin\, x\, sin\, x\, dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_{0}^{\pi} \sin^2 x\, dx\right\}$$

$$= \frac{1}{\pi}\int_{0}^{\pi}\left(\frac{1-\cos 2x}{2}\right) dx$$

$$= \frac{1}{2\pi}\left\{\int_{0}^{\pi} dx - \int_{0}^{\pi}\cos 2x\ \ dx\right\}$$

$$= \frac{1}{2\pi}\left\{[x]_0^\pi - \left[\frac{\sin 2x}{2}\right]_0^\pi\right\}$$

$$= \frac{1}{2\pi}\{\pi - 0\}$$

$$\boldsymbol{b_1 = \frac{1}{2}} \text{ ------------------------- (6)}$$

Substituting (2), (3), (4), (5) and (6) in (1), we have

$$f(x) = \frac{\frac{2}{\pi}}{2} + \sum_{n=2,4,6,\dots}^{\infty}\left(-\frac{2}{\pi(n^2-1)}\right)\cos nx + \frac{1}{2}\sin x$$

$$\boldsymbol{f(x) = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=2,4,6,\dots}^{\infty}\frac{1}{(n-1)(n+1)}\cos nx + \frac{1}{2}\sin x} \text{ ----------------(6)}$$

Put $x = 0$ is a point of continuity of $f(x)$, we have

From (6), $\sin 0 = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=2,4,6,\dots}^{\infty}\frac{1}{(n-1)(n+1)}\cos 0 + \frac{1}{2}\sin 0$

$$\frac{1}{\pi} - \frac{2}{\pi}\sum_{n=2,4,6,\dots}^{\infty}\frac{1}{(n-1)(n+1)} = 0$$

$$\frac{2}{\pi}\sum_{n=2,4,6,\dots}^{\infty}\frac{1}{(n-1)(n+1)} = \frac{1}{\pi}$$

$$\boldsymbol{\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots + \infty = \frac{1}{2}} \text{ ------------- } \textbf{proof (i)}$$

Put $x = \frac{\pi}{2}$ is a point of continuity of $f(x)$, we have

From (6), $\sin\frac{\pi}{2} = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=2,4,6,\dots}^{\infty} \frac{1}{(n-1)(n+1)} \cos\frac{n\pi}{2} + \frac{1}{2}\sin\frac{\pi}{2}$

$$\frac{1}{\pi} - \frac{2}{\pi}\sum_{n=2,4,6,\dots}^{\infty} \frac{1}{(n-1)(n+1)} \cos\frac{n\pi}{2} + \frac{1}{2} = 1$$

$$-\frac{2}{\pi}\sum_{n=2,4,6,\dots}^{\infty} \frac{1}{(n-1)(n+1)} \cos\frac{n\pi}{2} = 1 - \frac{1}{2} - \frac{1}{\pi}$$

$$-\frac{2}{\pi}\left[-\frac{1}{1.3} + \frac{1}{3.5} - \frac{1}{5.7} + \cdots\right] = \frac{2\pi-\pi-2}{2\pi}$$

$$\frac{2}{\pi}\left[\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \cdots\right] = \frac{\pi-2}{2\pi}$$

$$\mathbf{\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots + \infty = \frac{\pi-2}{4}}$$ **------------- proof (ii)**

6. Find the Fourier series of $f(x) = \begin{cases} k & if \ -1 < x < 0 \\ x & if \ \ \ 0 < x < 1 \end{cases}$. Hence find the sum of the series (i) $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots \infty$
(ii) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \infty$

Solution:

Given: $f(x) = \begin{cases} k & if \ -1 < x < 0 \\ x & if \ \ \ 0 < x < 1 \end{cases}$ in $(-1, 1)$

w.k.t. the Fourier Series expansion is

$$\boldsymbol{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x}$$ **--------------------- (1)**

To find a_0:

w.k.t. $a_0 = \int_{-1}^{1} f(x)dx$

$$= \int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx$$

$$= \int_{-1}^{0} k\,dx + \int_{0}^{1} x\,dx$$

$$= [kx]_{-1}^{0} + \left[\frac{x^2}{2}\right]_0^1$$

$$= [0 + k] + \left[\frac{1}{2} - 0\right]$$

$$\boldsymbol{a_0 = k + \frac{1}{2}}$$ **---------------------- (2)**

To find a_n:

w.k.t. $a_n = \int_{-1}^{1} f(x)\cos nx\, dx$

$$= \int_{-1}^{0} f(x)\cos nx\, dx + \int_{0}^{1} f(x)\cos nx\, dx$$

$$= \int_{-1}^{0} k\cos nx\, dx + \int_{0}^{1} x\cos n\pi x\, dx$$

Take $u = x$ $\qquad v = \cos n\pi x$

$u' = 1$ $\qquad v_1 = \frac{\sin n\pi x}{n\pi}$

$u'' = 0$ $\qquad v_2 = -\frac{\cos n\pi x}{n^2\pi^2}$

By using Bernoulli's Formula $\int uv dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$

$$= \left[\frac{k\sin n\pi x}{n\pi}\right]_{-1}^{0} + \left[x\left(\frac{\sin n\pi x}{n\pi}\right) - 1\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_{0}^{1}$$

$$= \left[\frac{k\sin(0)}{n\pi} - \frac{k\sin(-n\pi)}{n\pi}\right] + \left[1\left(\frac{\sin n\pi}{n\pi}\right) - 1\left(-\frac{\cos n\pi}{n^2\pi^2}\right) - 0\left(\frac{\sin(0)}{n\pi}\right) - \frac{\cos(0)}{n^2\pi^2}\right]$$

$$= \left[0 - 0 + 0 + \frac{(-1)^n}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right]$$

$$= \left[\frac{(-1)^n}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right]$$

$$= \left[\frac{(-1)^n - 1}{n^2\pi^2}\right]$$

$$\boldsymbol{a_n = \begin{cases} -\frac{2}{n^2\pi^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}} \quad \text{----------------------- } \mathbf{(3)}$$

To find b_n:

w.k.t. $b_n = \int_{-1}^{1} f(x)\sin n\pi x\, dx$

$$= \int_{-1}^{0} f(x)\sin n\pi x\, dx + \int_{0}^{1} f(x)\sin n\pi x\, dx$$

$$= \int_{-1}^{0} k\sin n\pi x\, dx + \int_{0}^{1} x\sin n\pi x\, dx$$

Take $u = x$ $\qquad v = \sin n\pi x$

$u' = 1 \qquad\qquad v_1 = -\frac{\cos n\pi x}{n\pi}$

$u'' = 0 \qquad\qquad v_2 = -\frac{\sin n\pi x}{n^2\pi^2}$

By using Bernoulli's Formula $\int uvdx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$

$$= \left[k\left(-\frac{\cos n\pi x}{n\pi}\right)\right]_{-1}^{0} + \left[x\left(-\frac{\cos n\pi x}{n\pi}\right) - 1\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_{0}^{1}$$

$$= \left[-\frac{k\cos(0)}{n\pi} - \frac{k\cos(-n\pi)}{n\pi}\right] + \left[1\left(-\frac{\cos n\pi}{n\pi}\right) - 1\left(-\frac{\sin n\pi}{n^2\pi^2}\right) - 0\left(-\frac{\cos(0)}{n\pi}\right) - \frac{\sin(0)}{n^2\pi^2}\right]$$

$$= \left[-\frac{k}{n\pi} - \frac{k(-1)^n}{n\pi} - \frac{(-1)^n}{n\pi} + 0 - 0 - 0\right]$$

$$\boldsymbol{b_n = -\frac{k}{n\pi}[1-(-1)^n] - \frac{(-1)^n}{n\pi}} \text{ ---------------- } \mathbf{(4)}$$

Substituting (2), (3) and (4) in (1), we have

$$f(x) = \frac{k+\frac{1}{2}}{2} + \sum_{n=1,3,5,\ldots}^{\infty}\left(-\frac{2}{n^2\pi^2}\right)\cos n\pi x + \sum_{n=1}^{\infty}\left(-\frac{k}{n\pi}[1-(-1)^n] - \frac{(-1)^n}{n\pi}\right)\sin n\pi x$$

$$\boldsymbol{f(x) = \frac{k}{2} + \frac{1}{4} + \sum_{n=1,3,5,\ldots}^{\infty}\left(-\frac{2}{n^2\pi^2}\right)\cos n\pi x + \sum_{n=1}^{\infty}\left(-\frac{k}{n\pi}[1-(-1)^n] - \frac{(-1)^n}{n\pi}\right)\sin n\pi x}$$

----------------- (5)

Put $x = 0$ is a point of discontinuity of $f(x)$, we have

$$\text{sum} = \lim_{h\to 0}\frac{f(0-h)+f(0+h)}{2}$$

$$= \lim_{h\to 0}\frac{k+h}{2}$$

$$= \frac{k}{2}$$

From (5), $\frac{k}{2} = \frac{k}{2} + \frac{1}{4} - \frac{2}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}\cos n\pi(0) - \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{k}{n}[1-(-1)^n] - \frac{(-1)^n}{n}\right)\sin n\pi(0)$

$$\frac{2}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2} = \frac{k}{2} + \frac{1}{4} - \frac{k}{2}$$

$$\frac{2}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2} = \frac{1}{4}$$

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots+\infty=\frac{\pi^2}{8}$$ ------------- **proof (i)**

Put $x=\frac{1}{2}$ is a point of continuity of $f(x)$, we have

$\text{sum} = f(x)$

$=\frac{1}{2}$

From (5),

$$\frac{1}{2}=\frac{k}{2}+\frac{1}{4}-\frac{2}{\pi^2}\sum_{n=1,3,5,\dots}^{\infty}\frac{1}{n^2}\cos n\pi\left(\frac{1}{2}\right)-\frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{k}{n}[1-(-1)^n]-\frac{(-1)^n}{n}\right)\sin n\pi\left(\frac{1}{2}\right)$$

$$\frac{1}{2}=\frac{k}{2}+\frac{1}{4}-\frac{2k-1}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)\sin n\pi\left(\frac{1}{2}\right)$$

$$-\frac{2k-1}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)\sin\frac{n\pi}{2}=\frac{1}{2}-\frac{k}{2}-\frac{1}{4}$$

$$-\left(\frac{2k-1}{\pi}\right)\left[\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\right]=\frac{2-2k-1}{4}$$

$$-\left(\frac{2k-1}{\pi}\right)\left[\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\right]=\frac{-2k+1}{4}$$

$$\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\infty=\frac{\pi}{4}$$ ------------- **proof (ii)**

7. Find the Fourier series of $f(x)=x^2$ in $(0,2l)$. Hence deduce that

(i) $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\infty=\frac{\pi^2}{6}$

(ii) $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\cdots+\infty=\frac{\pi^2}{12}$

(iii) $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots+\infty=\frac{\pi^2}{8}$

Solution:

Given: $f(x)= x^2$ in $(0,2l)$

w.k.t. the Fourier Series expansion is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}$$ --------------------- **(1)**

To find a_0:

w.k.t. $a_0=\frac{1}{l}\int_0^{2l}f(x)dx$

$$= \frac{1}{l}\int_0^{2l} x^2\, dx$$

$$= \frac{1}{l}\left[\frac{x^3}{3}\right]_0^{2l}$$

$$= \frac{1}{l}\left[\frac{8l^3}{3} - 0\right]$$

$$\boldsymbol{a_0} = \frac{\mathbf{8l^2}}{\mathbf{3}} \qquad \textbf{---------------------- (2)}$$

To find a_n:

w.k.t. $a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\, dx$

$$= \frac{1}{l}\int_0^{2l} x^2 \cos\frac{n\pi x}{l}\, dx$$

By using Bernoulli's Formula $\int uvdx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$

Take $u = x^2$ $\qquad$ $v = \cos\frac{n\pi x}{l}$

$u' = 2x$ $\qquad$ $v_1 = \frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}$

$u'' = 2$ $\qquad$ $v_2 = -\frac{\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}$

$v_3 = -\frac{\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}$

$$= \frac{1}{l}\left[x^2\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 2x\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + 2\left(-\frac{\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^{2l}$$

$$= \frac{1}{l}\left\{\left[4l^2\left(\frac{\sin\frac{n\pi(2l)}{l}}{\frac{n\pi}{l}}\right) + 4l\left(\frac{\cos\frac{n\pi(2l)}{l}}{\frac{n^2\pi^2}{l^2}}\right) - 2\left(\frac{\sin\frac{n\pi(2l)}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]\right.$$
$$\left. - \left[(0)\left(\frac{\sin\frac{n\pi(0)}{l}}{\frac{n\pi}{l}}\right) + 2(0)\left(\frac{\cos\frac{n\pi(0)}{l}}{\frac{n^2\pi^2}{l^2}}\right) - 2\left(\frac{\sin\frac{n\pi(0)}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]\right\}$$

$$= \frac{1}{l}\left\{4l^2\left(\frac{\sin 2n\pi}{\frac{n\pi}{l}}\right) + 4l\left(\frac{\cos 2n\pi}{\frac{n^2\pi^2}{l^2}}\right) - 2\left(\frac{\sin 2n\pi}{\frac{n^3\pi^3}{l^3}}\right)\right\}$$

$$= \frac{1}{l}\left\{4l^2\left(\frac{l(0)}{n\pi}\right) + 4l\left(\frac{l^2(-1)^2}{n^2\pi^2}\right) - 2\left(\frac{l^3(0)}{n^3\pi^3}\right)\right\}$$

$\because \sin n\pi = 0$ and $\cos n\pi = (-1)^n$

$$= \frac{1}{l}\left\{\frac{4l^3}{n^2\pi^2}\right\}$$

$\boldsymbol{a_n = \frac{4l^2}{n^2\pi^2}}$ **if $\boldsymbol{n \neq 0}$ ----------------------- (3)**

To find $\boldsymbol{b_n}$:

w.k.t. $b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$

$= \frac{1}{l}\int_0^{2l} x^2 \sin\frac{n\pi x}{l}\,dx$

By using Bernoulli's Formula $\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$

Take $u = x^2$ $\quad v = \sin\frac{n\pi x}{l}$

$u' = 2x$ $\quad v_1 = -\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}$

$u'' = 2$ $\quad v_2 = -\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}$

$v_3 = \frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}$

$$= \frac{1}{l}\left[x^2\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 2x\left(-\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + 2\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^{2l}$$

$$= \frac{1}{l}\left\{\left[4l^2\left(-\frac{\cos\frac{n\pi(2l)}{l}}{\frac{n\pi}{l}}\right) - 2(2l)\left(-\frac{\sin\frac{n\pi(2l)}{l}}{\frac{n^2\pi^2}{l^2}}\right) + 2\left(\frac{\cos\frac{n\pi(2l)}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]\right.$$
$$\left. - \left[(0)^2\left(-\frac{\cos\frac{n\pi(0)}{l}}{\frac{n\pi}{l}}\right) - 2(0)\left(-\frac{\sin\frac{n\pi(0)}{l}}{\frac{n^2\pi^2}{l^2}}\right) + 2\left(\frac{\cos\frac{n\pi(0)}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]\right\}$$

$$= \frac{1}{l}\left\{\left[4l^2\left(-\frac{\cos 2n\pi}{\frac{n\pi}{l}}\right) + 4l\left(-\frac{\sin 2n\pi}{\frac{n^2\pi^2}{l^2}}\right) + 2\left(\frac{\cos 2n\pi}{\frac{n^3\pi^3}{l^3}}\right)\right] - \left[2\left(\frac{\cos 0}{\frac{n^3\pi^3}{l^3}}\right)\right]\right\}$$

$$= \frac{1}{l}\left\{\left[4l^2\left(-\frac{l(-1)^2}{n\pi}\right) + 4l\left(-\frac{l^2(0)}{n^2\pi^2}\right) + 2\left(\frac{l^3(-1)^2}{n^3\pi^3}\right)\right] - \left[2\left(\frac{l^3}{n^3\pi^3}\right)\right]\right\}$$

$\because \sin n\pi = 0$ and $\cos n\pi = (-1)^n$

$$= \frac{1}{l}\left\{\left[4l^2\left(-\frac{l}{n\pi}\right) + 2\left(\frac{l^3}{n^3\pi^3}\right)\right] - \left[2\left(\frac{l^3}{n^3\pi^3}\right)\right]\right\}$$

$$= \frac{1}{l}\left\{\left[-\frac{4l^3}{n\pi} + \frac{2l^3}{n^3\pi^3}\right] - \left[\frac{2l^3}{n^3\pi^3}\right]\right\}$$

$$= \frac{1}{l}\left\{-\frac{4l^3}{n\pi}\right\}$$

$$\boldsymbol{b_n = -\frac{4l^2}{n\pi}} \text{------------------------- } \textbf{(4)}$$

Substituting (2), (3) and (4) in (1), we get

$$f(x) = \frac{\frac{8l^2}{3}}{2} + \sum_{n=1}^{\infty} \frac{4l^2}{n^2\pi^2}\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty}\left(-\frac{4l^2}{n\pi}\right)\sin\frac{n\pi x}{l}$$

$$\boldsymbol{= \frac{4l^2}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l} - \frac{4l^2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{l}} \text{ ------------------- } \textbf{(5)}$$

Put $x = 0$ is a point of discontinuity of the function $f(x)$, we have

$$f(x) = \frac{4l^2}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l} - \frac{4l^2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{l}$$

$$\frac{4l^2}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l} - \frac{4l^2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{l} = \lim_{h\to 0}\left[\frac{f(0-h)+f(0+h)}{2}\right]$$

$$\frac{4l^2}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos 0 - \frac{4l^2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin 0 = \lim_{h\to 0}\left[\frac{(-h+2l)^2+h^2}{2}\right]$$

$$\frac{4l^2}{\pi^2}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right) = 2l^2 - \frac{4l^2}{3}$$

$$\frac{4l^2}{\pi^2}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right) = \frac{2l^2}{3}$$

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\infty = \frac{\pi^2}{6}$$ **----------------- proof (i)**

Put $x = l$ is a point of continuity of the function $f(x)$, we have

$$f(x) = \frac{4l^2}{3}+\frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l}-\frac{4l^2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{l}$$

$$\frac{4l^2}{3}+\frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l}-\frac{4l^2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{l} = l^2$$

$$\frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi-\frac{4l^2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi = l^2-\frac{4l^2}{3}$$

$$\frac{4l^2}{\pi^2}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}(-1)^n\right) = -\frac{l^2}{3}$$

$$-\frac{1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\cdots+\infty = -\frac{\pi^2}{12}$$

$$-\left(\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\cdots+\infty\right) = -\frac{\pi^2}{12}$$

$$\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\cdots+\infty = \frac{\pi^2}{12}$$ **----------------- proof (ii)**

By adding proof (i) and (ii), we have

$$2\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots+\infty\right) = \frac{\pi^2}{6}+\frac{\pi^2}{12}$$

$$2\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots+\infty\right) = \frac{2\pi^2+\pi^2}{12}$$

$$2\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots+\infty\right) = \frac{\pi^2}{4}$$

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots+\infty = \frac{\pi^2}{8}$$ **------------------ proof (iii)**

8. Find the Fourier series expansion of $f(x) = x^2 + x$ in $(-2, 2)$. Hence find the sum of the series $1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\infty = \frac{\pi^2}{6}$.

Solution:

Given: $f(x) = x^2 + x$ in $(-2, 2)$

w.k.t. the Fourier Series expansion is

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{2}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{2}$$ **--------------------- (1)**

To find a_0:

w.k.t. $a_0 = \frac{1}{2}\int_{-2}^{2} f(x)dx$

$= \frac{1}{2}\int_{-2}^{2}(x^2 + x)\ dx$

$= \frac{1}{2}\int_{0}^{2}(x^2 + x)\ dx$

$= \frac{1}{2}\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_{-2}^{2}$

$= \frac{1}{2}\left[\left(\frac{8}{3} + \frac{4}{2}\right) - \left(-\frac{8}{3} + \frac{4}{2}\right)\right]$

$\boldsymbol{a_0 = \frac{8}{3}}$ **---------------------- (2)**

To find a_n:

w.k.t. $a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}dx$

$= \frac{1}{2}\int_{-2}^{2}(x^2 + x)\ \cos\frac{n\pi x}{2}dx$

$= \frac{1}{2}\left\{\int_{-2}^{2} x^2\ \cos\frac{n\pi x}{2}dx + \int_{-2}^{2} x\ \cos\frac{n\pi x}{2}dx\right\}$

By using Bernoulli's Formula $\int uvdx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$

Take $u = x^2$	$v = \cos\frac{n\pi x}{2}$	$u = x$
$u' = 2x$	$v_1 = \frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}$	$u' = 1$
$u'' = 2$	$v_2 = -\frac{\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}$	$u'' = 0$
	$v_3 = -\frac{\sin\frac{n\pi x}{2}}{\frac{n^3\pi^3}{8}}$	

$$= \frac{1}{2}\left\{\left[x^2\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - 2x\left(-\frac{\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right) + 2\left(-\frac{\sin\frac{n\pi x}{2}}{\frac{n^3\pi^3}{8}}\right)\right]_{-2}^{2} + \left[x\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - 1\left(-\frac{\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right]_{-2}^{2}\right\}$$

$$= \frac{1}{2}\left\{\left[4\left(\frac{\sin\frac{n\pi(2)}{2}}{\frac{n\pi}{2}}\right) + 4\left(\frac{\cos\frac{n\pi(2)}{2}}{\frac{n^2\pi^2}{4}}\right) - 2\left(\frac{\sin\frac{n\pi(2)}{2}}{\frac{n^3\pi^3}{8}}\right)\right] - \left[4\left(\frac{\sin\frac{n\pi(-2)}{2}}{\frac{n\pi}{2}}\right) + 4\left(-\frac{\cos\frac{n\pi(-2)}{2}}{\frac{n^2\pi^2}{4}}\right) - 2\left(\frac{\sin\frac{n\pi(-2)}{2}}{\frac{n^3\pi^3}{8}}\right)\right] + \left[2\left(\frac{\sin\frac{n\pi 2}{2}}{\frac{n\pi}{2}}\right) - 1\left(-\frac{\cos\frac{n\pi 2}{2}}{\frac{n^2\pi^2}{4}}\right)\right] - \left[-2\left(\frac{\sin\frac{n\pi(-2)}{2}}{\frac{n\pi}{2}}\right) - 1\left(-\frac{\cos\frac{n\pi(-2)}{2}}{\frac{n^2\pi^2}{4}}\right)\right]\right\}$$

$$= \frac{1}{2}\left\{\left[4\left(\frac{\sin n\pi}{\frac{n\pi}{2}}\right) + 4\left(\frac{\cos n\pi}{\frac{n^2\pi^2}{4}}\right) - 2\left(\frac{\sin n\pi}{\frac{n^3\pi^3}{8}}\right)\right] - \left[4\left(\frac{\sin(-n\pi)}{\frac{n\pi}{2}}\right) - 4\left(\frac{\cos(-n\pi)}{\frac{n^2\pi^2}{4}}\right) - 2\left(\frac{\sin(-n\pi)}{\frac{n^3\pi^3}{8}}\right)\right] + \left[2\left(\frac{\sin n\pi}{\frac{n\pi}{2}}\right) - 1\left(-\frac{\cos n\pi}{\frac{n^2\pi^2}{4}}\right)\right] - \left[-2\left(\frac{\sin(-n\pi)}{\frac{n\pi}{2}}\right) - 1\left(-\frac{\cos(-n\pi)}{\frac{n^2\pi^2}{4}}\right)\right]\right\}$$

$$= \frac{1}{2}\left\{\left[4\left(\frac{0}{\frac{n\pi}{2}}\right) + 4\left(\frac{(-1)^n}{\frac{n^2\pi^2}{4}}\right) - 2\left(\frac{0}{\frac{n^3\pi^3}{8}}\right)\right] - \left[4\left(\frac{0}{\frac{n\pi}{2}}\right) - 4\left(\frac{(-1)^n}{\frac{n^2\pi^2}{4}}\right) - 2\left(\frac{0}{\frac{n^3\pi^3}{8}}\right)\right] + \left[2\left(\frac{0}{\frac{n\pi}{2}}\right) - 1\left(-\frac{(-1)^n}{\frac{n^2\pi^2}{4}}\right)\right] - \left[-2\left(\frac{0}{\frac{n\pi}{2}}\right) - 1\left(-\frac{(-1)^n}{\frac{n^2\pi^2}{4}}\right)\right]\right\}$$

$\because \sin n\pi = 0$ and $\cos n\pi = (-1)^n$

$$= \frac{1}{2}\left\{\left[4\left(\frac{(-1)^n}{\frac{n^2\pi^2}{4}}\right)\right] + \left[4\left(\frac{(-1)^n}{\frac{n^2\pi^2}{4}}\right)\right] + \left[-1\left(-\frac{(-1)^n}{\frac{n^2\pi^2}{4}}\right)\right] - \left[-1\left(-\frac{(-1)^n}{\frac{n^2\pi^2}{4}}\right)\right]\right\}$$

$$= \frac{1}{2}\left\{\left[16\left(\frac{(-1)^n}{n^2\pi^2}\right)\right] + \left[16\left(\frac{(-1)^n}{n^2\pi^2}\right)\right] + \left[4\left(\frac{(-1)^n}{n^2\pi^2}\right)\right] - \left[4\left(\frac{(-1)^n}{n^2\pi^2}\right)\right]\right\}$$

$\boldsymbol{a_n = \frac{16(-1)^n}{n^2\pi^2}}$ **if $\boldsymbol{n \neq 0}$ ----------------------- (3)**

To find $\boldsymbol{b_n}$:

w.k.t. $b_n = \frac{1}{2}\int_{-2}^{2} f(x) \sin\frac{n\pi x}{2} dx$

$$= \frac{1}{2}\int_{-2}^{2}(x^2 + x) \sin\frac{n\pi x}{2} dx$$

$$= \frac{1}{2}\left\{\int_{-2}^{2} x^2 \sin\frac{n\pi x}{2} dx + \int_{-2}^{2} x \sin\frac{n\pi x}{2} dx\right\}$$

By using Bernoulli's Formula $\int uvdx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$

Take $u = x^2$	$v = \sin\frac{n\pi x}{2}$	$u = x$
$u' = 2x$	$v_1 = -\frac{\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}$	$u' = 1$
$u'' = 2$	$v_2 = -\frac{\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}$	$u'' = 0$
	$v_3 = \frac{\cos\frac{n\pi x}{2}}{\frac{n^3\pi^3}{8}}$	

$$= \frac{1}{2}\left\{\left[x^2\left(-\frac{\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - 2x\left(-\frac{\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right) + 2\left(\frac{\cos\frac{n\pi x}{2}}{\frac{n^3\pi^3}{8}}\right)\right]_{-2}^{2} + \left[x\left(-\frac{\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - 1\left(-\frac{\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right]_{-2}^{2}\right\}$$

$$=\frac{1}{2}\left\{\left[4\left(-\frac{\cos\frac{n\pi(2)}{2}}{\frac{n\pi}{2}}\right)+4\left(-\frac{\sin\frac{n\pi(2)}{2}}{\frac{n^2\pi^2}{4}}\right)-2\left(\frac{\cos\frac{n\pi(2)}{2}}{\frac{n^3\pi^3}{8}}\right)\right]\right.$$

$$-\left[4\left(-\frac{\cos\frac{n\pi(-2)}{2}}{\frac{n\pi}{2}}\right)+4\left(-\frac{\sin\frac{n\pi(-2)}{2}}{\frac{n^2\pi^2}{4}}\right)-2\left(\frac{\cos\frac{n\pi(-2)}{2}}{\frac{n^3\pi^3}{8}}\right)\right]$$

$$\left.+\left[2\left(-\frac{\cos\frac{n\pi2}{2}}{\frac{n\pi}{2}}\right)-1\left(-\frac{\sin\frac{n\pi2}{2}}{\frac{n^2\pi^2}{4}}\right)\right]-\left[-2\left(-\frac{\cos\frac{n\pi(-2)}{2}}{\frac{n\pi}{2}}\right)-1\left(-\frac{\sin\frac{n\pi(-2)}{2}}{\frac{n^2\pi^2}{4}}\right)\right]\right\}$$

$$=\frac{1}{2}\left\{\left[4\left(-\frac{\cos n\pi}{\frac{n\pi}{2}}\right)+4\left(-\frac{\sin n\pi}{\frac{n^2\pi^2}{4}}\right)-2\left(\frac{\cos n\pi}{\frac{n^3\pi^3}{8}}\right)\right]\right.$$

$$-\left[4\left(-\frac{\cos(-n\pi)}{\frac{n\pi}{2}}\right)-4\left(-\frac{\sin(-n\pi)}{\frac{n^2\pi^2}{4}}\right)-2\left(\frac{\cos(-n\pi)}{\frac{n^3\pi^3}{8}}\right)\right]$$

$$\left.+\left[2\left(-\frac{\cos n\pi}{\frac{n\pi}{2}}\right)-1\left(-\frac{\sin n\pi}{\frac{n^2\pi^2}{4}}\right)\right]-\left[-2\left(-\frac{\cos(-n\pi)}{\frac{n\pi}{2}}\right)-1\left(-\frac{\sin(-n\pi)}{\frac{n^2\pi^2}{4}}\right)\right]\right\}$$

$$=\frac{1}{2}\left\{\left[4\left(-\frac{(-1)^n}{\frac{n\pi}{2}}\right)+4\left(-\frac{0}{\frac{n^2\pi^2}{4}}\right)-2\left(\frac{(-1)^n}{\frac{n^3\pi^3}{8}}\right)\right]\right.$$

$$-\left[4\left(-\frac{(-1)^n}{\frac{n\pi}{2}}\right)-4\left(-\frac{0}{\frac{n^2\pi^2}{4}}\right)-2\left(\frac{(-1)^n}{\frac{n^3\pi^3}{8}}\right)\right]+\left[2\left(-\frac{(-1)^n}{\frac{n\pi}{2}}\right)-1\left(-\frac{0}{\frac{n^2\pi^2}{4}}\right)\right]$$

$$\left.-\left[-2\left(-\frac{(-1)^n}{\frac{n\pi}{2}}\right)-1\left(-\frac{0}{\frac{n^2\pi^2}{4}}\right)\right]\right\}$$

$\because \sin n\pi = 0$ and $\cos n\pi = (-1)^n$

$$=\frac{1}{2}\left\{\left[4\left(-\frac{(-1)^n}{\frac{n\pi}{2}}\right)-2\left(\frac{(-1)^n}{\frac{n^3\pi^3}{8}}\right)\right]-\left[4\left(-\frac{(-1)^n}{\frac{n\pi}{2}}\right)-2\left(\frac{(-1)^n}{\frac{n^3\pi^3}{8}}\right)\right]+\left[2\left(-\frac{(-1)^n}{\frac{n\pi}{2}}\right)\right]\right.$$

$$\left.-\left[-2\left(-\frac{(-1)^n}{\frac{n\pi}{2}}\right)\right]\right\}$$

$$= \frac{1}{2}\left\{\left[8\left(-\frac{(-1)^n}{n\pi}\right) - 16\left(\frac{(-1)^n}{n^3\pi^3}\right)\right] - \left[8\left(-\frac{(-1)^n}{n\pi}\right) - 16\left(\frac{(-1)^n}{n^3\pi^3}\right)\right] + \left[4\left(-\frac{(-1)^n}{n\pi}\right)\right] - \left[-4\left(-\frac{(-1)^n}{n\pi}\right)\right]\right\}$$

$$= \frac{1}{2}\left\{\left[8\left(-\frac{(-1)^n}{n\pi}\right) - 16\left(\frac{(-1)^n}{n^3\pi^3}\right)\right] + \left[8\left(\frac{(-1)^n}{n\pi}\right) + 16\left(\frac{(-1)^n}{n^3\pi^3}\right)\right] + \left[4\left(-\frac{(-1)^n}{n\pi}\right)\right] - \left[4\left(\frac{(-1)^n}{n\pi}\right)\right]\right\}$$

$$\boldsymbol{b_n = -\frac{4(-1)^n}{n\pi} \ \text{if } n \neq 0} \quad \textbf{----------------------- (4)}$$

Substituting (2), (3) and (4) in (1), we get

$$f(x) = \frac{\frac{8}{3}}{2} + \sum_{n=1}^{\infty} \frac{16(-1)^n}{n^2\pi^2} \cos\frac{n\pi x}{2} + \sum_{n=1}^{\infty}\left(-\frac{4(-1)^n}{n\pi}\right)\sin\frac{n\pi x}{2}$$

$$\boldsymbol{= \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\frac{n\pi x}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin\frac{n\pi x}{2}} \quad \textbf{------------------- (5)}$$

Put $x = 2$ is a point of discontinuity of the function $f(x)$, we have

$$f(x) = \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\frac{n\pi x}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin\frac{n\pi x}{2}$$

$$\frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\frac{n\pi(2)}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin\frac{n\pi(2)}{2} = \lim_{h\to 0}\left[\frac{\{(2-h)^2+(2-h)\}+\{(2+h-4)^2+(2+h-4)\}}{2}\right]$$

$$\frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin n\pi = 4$$

$$\frac{16}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} (-1)^n - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (0) = 4 - \frac{4}{3}$$

$$\frac{16}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} (-1)^n = \frac{8}{3}$$

$$\boldsymbol{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \infty = \frac{\pi^2}{6}}$$

9. Find the Fourier series of the period 2π for the function $f(x) = x\cos x$ in $0 < x < 2\pi$.

Solution:

Given: $f(x) = x\cos x$ in $(0, 2\pi)$

w.k.t. the Fourier Series expansion is

$$\boldsymbol{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx} \quad \text{--------------------- (1)}$$

To find a_0:

w.k.t. $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$

$$= \frac{1}{\pi}\left\{\int_0^{2\pi} x\cos x\, dx\right\}$$

$$= \frac{1}{\pi}[x(\sin x) - 1(\cos x)]_0^{2\pi}$$

$$= \frac{1}{\pi}[(2\pi\sin 2\pi - \cos 2\pi) - (0 - \cos 0)]$$

$$= \frac{1}{\pi}[(0-1) - (0-1)]$$

$$\boldsymbol{a_0 = 0} \quad \text{---------------------- (2)}$$

To find a_n:

w.k.t. $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$

$$= \frac{1}{\pi}\left\{\int_0^{2\pi} f(x)\cos nx\, dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^{2\pi} x\ \cos x\cos nx\, dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^{2\pi} x\left[\frac{cos(n+1)x + \cos(n-1)x}{2}\right] dx\right\}$$

$$= \frac{1}{2\pi}\left\{\int_0^{2\pi} x\ cos(n+1)x\ \ dx + \int_0^{\pi} x\ cos(n-1)x\ \ dx\right\}$$

$$= \frac{1}{2\pi}\left\{\left[\frac{x\sin(n+1)x}{n+1} + \frac{\cos(n+1)x}{(n+1)^2}\right]_0^{2\pi} + \left[\frac{x\sin(n-1)x}{n-1} + \frac{\cos(n-1)x}{(n-1)^2}\right]_0^{2\pi}\right\}$$

$$= \frac{1}{2\pi}\left\{\begin{array}{c}\left[\frac{2\pi\sin 2(n+1)\pi}{n+1} + \frac{\cos 2(n+1)\pi}{(n+1)^2} - \frac{0\sin 0}{n+1} - \frac{\cos 0}{(n+1)^2}\right] + \\ \left[\frac{2\pi\sin 2(n-1)\pi}{n-1} + \frac{\cos 2(n-1)\pi}{(n-1)^2} - \frac{0\sin 0}{n-1} - \frac{\cos 0}{(n-1)^2}\right]\end{array}\right\}$$

$$= \frac{1}{2\pi}\left\{\left[\frac{\cos 2(n+1)\pi}{(n+1)^2} - \frac{1}{(n+1)^2}\right] + \left[\frac{\cos 2(n-1)\pi}{(n-1)^2} - \frac{1}{(n-1)^2}\right]\right\}$$

$$= \frac{1}{2\pi}\left\{\left[\frac{1}{(n+1)^2} - \frac{1}{(n+1)^2}\right] + \left[\frac{1}{(n-1)^2} - \frac{1}{(n-1)^2}\right]\right\}$$

$\boldsymbol{a_n = 0}$ **----------------------- (3)**

To find b_n:

w.k.t. $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$

$$= \frac{1}{\pi}\left\{\int_0^{2\pi} f(x)\sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^{2\pi} x\ \cos x\sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^{2\pi} x\left[\frac{sin(n+1)x + \sin(n-1)x}{2}\right]dx\right\}$$

$$= \frac{1}{2\pi}\left\{\int_0^{2\pi} x\ sin(n+1)x\ \ dx + \int_0^{\pi} x\ sin(n-1)x\ \ dx\right\}$$

$$= \frac{1}{2\pi}\left\{\left[\frac{x(-\cos(n+1)x)}{n+1} + \frac{\sin(n+1)x}{(n+1)^2}\right]_0^{2\pi} + \left[\frac{x\,(-\cos(n-1)x)}{n-1} + \frac{\sin(n-1)x}{(n-1)^2}\right]_0^{2\pi}\right\}$$

$$= \frac{1}{2\pi}\left\{\begin{array}{c}\left[\frac{2\pi(-\cos 2(n+1)\pi)}{n+1} + \frac{\sin 2(n+1)\pi}{(n+1)^2} - \frac{0(-\cos 0)}{n+1} - \frac{\sin 0}{(n+1)^2}\right] + \\ \left[\frac{2\pi(-\cos 2(n-1)\pi)}{n-1} + \frac{\sin 2(n-1)\pi}{(n-1)^2} - \frac{0(-\cos 0)}{n-1} - \frac{\sin 0}{(n-1)^2}\right]\end{array}\right\}$$

$$= \frac{1}{2\pi}\left\{\left[-\frac{2\pi\cos 2(n+1)\pi}{n+1}\right] + \left[-\frac{2\pi\cos 2(n-1)\pi}{n-1}\right]\right\}$$

$$= \left\{\left[-\frac{1}{n+1}\right] + \left[-\frac{1}{n-1}\right]\right\}$$

$\boldsymbol{b_n = -\frac{2n}{n^2-1}}$ **----------------------- (4)**

Put $n = 1$ in (2), we get

(2) $\Rightarrow a_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos x\,dx$

$$= \frac{1}{\pi}\left\{\int_0^{2\pi} f(x)\cos x\, dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^{2\pi} x\ \cos x \cos x\, dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^{2\pi} x\, cos^2\, x\, dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^{2\pi} x\,(1+\cos 2x)\, dx\right\}$$

$$= \frac{1}{2\pi}\left[\frac{x^2}{2} + x\left(\frac{\sin 2x}{2}\right) + \frac{\cos 2x}{4}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left\{\left[2\pi^2 + 2\pi\left(\frac{\sin 4\pi}{2}\right) + \frac{\cos 4\pi}{4}\right] - \left[0 + 0\left(\frac{\sin 0}{2}\right) + \frac{\cos 0}{4}\right]\right\}$$

$$= \frac{1}{2\pi}\left\{\left[2\pi^2 - \frac{1}{4}\right] - \left[\frac{1}{4}\right]\right\}$$

$\boldsymbol{a_1 = \pi}$ **----------------------- (5)**

Put $n = 1$ in (3), we get

$$(3) \Rightarrow b_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin x\, dx$$

$$= \frac{1}{\pi}\left\{\int_0^{2\pi} f(x)\sin x\, dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^{2\pi} x\ \cos x \sin x\, dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^{2\pi} \frac{x\sin 2x}{2}\, dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^{2\pi} x\sin 2x\, dx\right\}$$

$$= \frac{1}{2\pi}\left[x\left(-\frac{\cos 2x}{2}\right) + \frac{\sin 2x}{4}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left\{\left[2\pi\left(-\frac{\cos 4\pi}{2}\right) + \frac{\sin 4\pi}{4}\right] - \left[2\pi\left(-\frac{\cos 0}{2}\right) + \frac{\sin 0}{4}\right]\right\}$$

$$= \frac{1}{2\pi}\{-\pi - \pi\}$$

$\boldsymbol{b_1 = -\frac{1}{2}}$ **----------------------- (6)**

Substituting (2), (3), (4), (5) and (6) in (1), we have

$$f(x) = \frac{0}{2} + \pi\cos x + \sum_{n=2,3,5,\ldots}^{\infty}(0)\cos nx - \frac{1}{2}\sin x + \sum_{n=1}^{\infty}\left(-\frac{2n}{n^2-1}\right)\sin nx$$

$$\boldsymbol{f(x) = \pi\cos x - \frac{1}{2}\sin x - 2\sum_{n=1}^{\infty}\left(\frac{n}{n^2-1}\right)\sin nx}$$

TRY YOURSELF

1. Find the Fourier series expansion of $f(x) = x(1-x)(2-x)$ in $(0,2)$. Deduce the sum of the series $1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots\infty = \frac{\pi^3}{32}$.

2. Find the Fourier series for $f(x)$ of period $2l$ and defined as follow $f(x) = \begin{cases} l-x & if\ 0 < x \le l \\ 0 & if\ l \le x < 2l \end{cases}$. Hence deduce the sum of infinity of the series $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$.

3. Find the Fourier series of the period 2π for the function $f(x) = x\ sin\ x$ in $0 < x < 2\pi$.

4. Find the Fourier series for $f(x) = e^{ax}$ in $(0, 2\pi)$.

5. Find the Fourier series of $f(x) = \begin{cases} 1 & if\ \ 0 < x < \pi \\ 2 & if\ \pi < x < 2\pi \end{cases}$.

Even and Odd Functions

Even and Odd function cases arises only when the function is defined in $(-l\,, l)$ and $(-\pi, \pi)$.

Definition:

Even: A function $f(x)$ is said to be even if $f(-x) = f(x)$

Odd: A function $f(x)$ is said to be odd if $f(-x) = -f(x)$ (or) $f(x) = -f(-x)$

Note:

- The Fourier function $f(x)$ is even if $b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

- The Fourier function $f(x)$ is odd if $a_0 = a_n = 0$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

10. Find the Fourier series of the period 2π for the function $f(x) = |cos\, x|$ in $-\pi < x < \pi$.

Solution:

Given: $f(x) = |cos\, x|$ in $(-\pi, \pi)$

Here $\boldsymbol{f}(-\boldsymbol{x}) = |\cos(-x)|$

$= |\cos x|$

$= \boldsymbol{f}(\boldsymbol{x})$ is an even function

w.k.t. the even function of the Fourier Series expansion is

$\boldsymbol{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx}$ **-------------------- (1)**

To find a_0:

w.k.t. $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$

$= \frac{2}{\pi}\left\{\int_0^{\frac{\pi}{2}} f(x)\,dx + \int_{\frac{\pi}{2}}^{\pi} f(x)\,dx\right\}$

$= \frac{2}{\pi}\left\{\int_0^{\frac{\pi}{2}} \cos x\,dx - \int_{\frac{\pi}{2}}^{\pi} \cos x\,dx\right\}$

$= \frac{2}{\pi}\left\{[\sin x]_0^{\frac{\pi}{2}} - [\sin x]_{\frac{\pi}{2}}^{\pi}\right\}$

$= \frac{2}{\pi}\left\{\sin\frac{\pi}{2} - \sin 0 - \sin\pi + \sin\frac{\pi}{2}\right\}$

$= \frac{2}{\pi}[2]$

$\boldsymbol{a_0 = \frac{4}{\pi}}$ **---------------------- (2)**

To find a_n:

w.k.t. $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

$= \frac{2}{\pi}\left\{\int_0^{\frac{\pi}{2}} f(x)\cos nx\,dx + \int_{\frac{\pi}{2}}^{\pi} f(x)\cos nx\,dx\right\}$

$= \frac{2}{\pi}\left\{\int_0^{\frac{\pi}{2}} \cos x\cos nx\,dx - \int_{\frac{\pi}{2}}^{\pi} \cos x\cos nx\,dx\right\}$

$$= \frac{2}{\pi}\left\{\int_0^{\frac{\pi}{2}} x\left[\frac{cos(n+1)x+\cos(n-1)x}{2}\right]dx - \int_{\frac{\pi}{2}}^{\pi} x\left[\frac{cos(n+1)x+\cos(n-1)x}{2}\right]dx\right\}$$

$$= \frac{2}{\pi}\left\{\int_0^{\frac{\pi}{2}} cos(n+1)x\ dx + \int_0^{\frac{\pi}{2}} cos(n-1)x\ dx - \int_{\frac{\pi}{2}}^{\pi} cos(n+1)x\ dx - \int_{\frac{\pi}{2}}^{\pi} cos(n-1)x\ dx\right\}$$

$$= \frac{2}{\pi}\left\{\left[\frac{\sin(n+1)x}{n+1}\right]_0^{\frac{\pi}{2}} + \left[\frac{\sin(n-1)x}{n-1}\right]_0^{\frac{\pi}{2}} - \left[\frac{\sin(n+1)x}{n+1}\right]_{\frac{\pi}{2}}^{\pi} - \left[\frac{\sin(n-1)x}{n-1}\right]_{\frac{\pi}{2}}^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{\sin\frac{(n+1)\pi}{2}}{n+1} - \frac{\sin 0}{n+1} + \frac{\sin\frac{(n-1)\pi}{2}}{n-1} - \frac{\sin 0}{n-1} - \frac{\sin(n+1)\pi}{n+1} + \frac{\sin\frac{(n+1)\pi}{2}}{n+1} - \frac{\sin(n-1)\pi}{n-1} + \frac{\sin\frac{(n-1)\pi}{2}}{n-1}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{\sin\frac{(n+1)\pi}{2}}{n+1} + \frac{\sin\frac{(n-1)\pi}{2}}{n-1} + \frac{\sin\frac{(n+1)\pi}{2}}{n+1} + \frac{\sin\frac{(n-1)\pi}{2}}{n-1}\right\}$$

$$\because \sin(n \pm 1) = \sin\frac{n\pi}{2}\cos\frac{\pi}{2} \pm \cos\frac{n\pi}{2}\sin\frac{\pi}{2} = \pm\cos\frac{n\pi}{2}$$

$$= \frac{1}{2\pi}\left\{\left[\frac{1}{n+1} - \frac{1}{n-1}\right]\cos\frac{n\pi}{2}\right\}$$

$$\boldsymbol{a_n = -\frac{4}{\pi(n^2-1)}\ cos\frac{n\pi}{2}} \text{ ------------------------ } \mathbf{(3)}$$

Substituting (2) and (3)in (1), we have

$$f(x) = \frac{\frac{4}{\pi}}{2} + \sum_{n=1}^{\infty}\left(-\frac{\mathbf{4}}{\boldsymbol{\pi(n^2-1)}}\ \boldsymbol{cos\frac{n\pi}{2}}\right)\cos nx$$

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=2,4,6,\dots}^{\infty}\frac{1}{(n^2-1)}\ cos\frac{n\pi}{2}\cos nx$$

when n is odd $\cos\frac{n\pi}{2} = 0$

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(n^2-1)}\ cos\frac{n\pi}{2}\cos nx$$

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\ cosn\pi\cos 2nx$$

$$\boldsymbol{f(x) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{4n^2-1}\ \cos 2nx}$$

11. Find the Fourier series exapansion of $f(x) = \sin ax$ in $(-l, l)$.

Solution:

Given: $f(x) = \sin ax$ in $(-l, l)$

Here $\boldsymbol{f(-x)} = \sin a(-x)$

$= -\sin ax$

$= \boldsymbol{-f(x)}$ is an odd function

w.k.t. the odd function of the Fourier Series expansion is

$$\boldsymbol{f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}} \quad \text{---------------------} \ \textbf{(1)}$$

To find b_n:

w.k.t. $b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l} dx$

$$= \frac{2}{l}\int_0^l \sin ax \sin \frac{n\pi x}{l} dx$$

$$= \frac{2}{l}\left\{\int_0^l \frac{cos\left(\frac{n\pi}{l}-a\right)x - \cos\left(\frac{n\pi}{l}+a\right)x}{2} dx\right\}$$

$$= \frac{1}{l}\left[\frac{\sin\left(\frac{n\pi}{l}-a\right)x}{\frac{n\pi}{l}-a} - \frac{\sin\left(\frac{n\pi}{l}+a\right)x}{\frac{n\pi}{l}+a}\right]_0^l$$

$$= \frac{1}{l}\left[\frac{\sin\left(\frac{n\pi}{l}-a\right)l}{\frac{n\pi}{l}-a} - \frac{\sin\left(\frac{n\pi}{l}+a\right)l}{\frac{n\pi}{l}+a}\right]$$

$$= \frac{1}{n\pi - al}[\sin(n\pi - al)] - \frac{1}{n\pi + al}[\sin(n\pi + al)]$$

$$= \frac{1}{n\pi - al}[(-1)^n \sin al] - \frac{1}{n\pi + al}[(-1)^n \sin al]$$

$$\boldsymbol{b_n = \frac{2\pi(-1)^n n}{n^2\pi^2 - a^2l^2} \ sin \ al} \quad \text{-----------------------} \ \textbf{(2)}$$

Substituting (2) in (1), we have

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{2\pi(-1)^n n}{n^2\pi^2 - a^2l^2} \ sin \ al\right)\sin \frac{n\pi x}{l}$$

$$\boldsymbol{f(x) = 2\pi \sin al \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2\pi^2 - a^2l^2} \sin \frac{n\pi x}{l}}$$

TRY YOURSELF

1. Find the Fourier series expansion of $f(x) = \begin{cases} 1+\frac{2x}{l} & if\ -l \le x \le 0 \\ 1-\frac{2x}{l} & if\ \ 0 \le x \le l \end{cases}$ .Hence deduce the sum of the series $1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots+\infty = \frac{\pi^2}{8}$.

2. Find the Fourier series expansion of period 2π for the function $f(x) = \begin{cases} x(\pi - x) & in -\pi \le x \le 0 \\ x(\pi + x) & in\ \ 0 \le x \le \pi \end{cases}$.

3. Find the Fourier series expansion of period 2π for the function $f(x) = \sqrt{1-\cos x}$ in $-\pi < x < \pi$.

4. Find the Fourier series expansion of period 2π for the function $f(x) = \sinh \alpha x$ in $(-\pi, \pi)$.

5. Find the Fourier series expansion of $f(x)$in $(-2, 2)$ which is defined as follows:

$$f(x) = \begin{cases} 0 & in\ (-2,-1) \\ x + x^2 & in\ (-1,0) \\ x - x^2 & in\ (0,1) \\ 0 & in\ \ (1,2) \end{cases}.$$

UNIT II

HALF RANGE FOURIER SERIES

In many Engineering problems it is required to expand a function $f(x)$ in the range $(0,\pi)$in a Fourier series of period 2π or in the range $(0,l)$ in Fourier series of period $2l$. If it is required to expand $f(x)$ in the interval $(0,l)$, then it is immaterial what the function may be outside the range $0<x<l$. We are free to choose it arbitrarily in the interval $(-l,0)$.

If we extend the function $f(x)$ by reflecting it in the y axis so the $f(-x))=f(x)$, then the extended function is even for which $b_n=0$. The Fourier expansion of $f(x)$ will contain only $cosine$ terms.

If we extend the function $f(x)$ by reflecting it in the origin so that $f(-x)=-f(x)$, then the extended function is odd for which $a_0=a_n=0$. The Fourier expansion of $f(x)$ will contain only $sine$ terms.

Hence a function $f(x)$ defined over the interval $0<x<l$ is capable of two distinct half range series. (i) Sine series

(ii) Cosine series

Half range of Cosine series in $(0,l)$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}$$

Where Euler's formula is

$$a_0=\frac{2}{l}\int_0^l f(x)\ dx$$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\ dx$$

Half range of Cosine series in $(0,\pi)$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx$$

Where Euler's formula is

$$a_0=\frac{2}{\pi}\int_0^{\pi} f(x)\ dx$$

$$a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\ dx$$

Half range of Sine series in $(0,l)$

$$f(x)=\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}$$

Where Euler's formula is

$$b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\ dx$$

Half range of Sine series in $(0,\pi)$

$$f(x)=\sum_{n=1}^{\infty} b_n\sin nx$$

Where Euler's formula is

$$b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\ dx$$

1. Find the half-range sine series of $f(x) = x$ in $(0, l)$.

Solution:

Given: $f(x) = x$ in $(0, l)$

w.k.t. the Fourier half-range sine Series expansion is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \text{ --------------------- (1)}$$

To find b_n:

w.k.t. $b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$

$$= \frac{2}{l}\int_0^l x\sin\frac{n\pi x}{l}dx$$

Take $u = x$ $\quad v = \sin\frac{n\pi x}{l}$

$u' = 1$ $\quad v_1 = -\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}$

$u'' = 0$ $\quad v_2 = -\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}$

$$= \frac{2}{l}\left[x\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 1\left(-\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$= \frac{2}{l}\left\{\left[l\left(-\frac{\cos n\pi}{\frac{n\pi}{l}}\right) - 1\left(-\frac{\sin n\pi}{\frac{n^2\pi^2}{l^2}}\right)\right] - \left[(0)\left(-\frac{\cos 0}{\frac{n\pi}{l}}\right) - 1\left(-\frac{\sin 0}{\frac{n^2\pi^2}{l^2}}\right)\right]\right\}$$

$$= \frac{2}{l}\left\{\left[\left(-\frac{l^2(-1)^n}{n\pi}\right) - 1\left(-\frac{l^2(0)}{n^2\pi^2}\right)\right]\right\}$$

$$b_n = -\frac{2l(-1)^n}{n\pi} \text{ ----------------------- (2)}$$

Substituting (2) in (1), we have

$$f(x) = \sum_{n=1}^{\infty}\left(-\frac{2l(-1)^n}{n\pi}\right)\sin\frac{n\pi x}{l}$$

$$f(x) = \frac{2l}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l}$$

2. Find the half-range (i) cosine series and (ii) sine series for $f(x) = x^2$in $(0, \pi)$.

Solution:

Given: $f(x) = x^2$ in $(0, \pi)$

(i) w.k.t. the Fourier half-range cosine Series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \text{ --------------------- (1)}$$

To find a_0:

w.k.t. $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\, dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left\{\left[\frac{\pi^3}{3} - 0\right]\right\}$$

$$a_0 = -\frac{2\pi^2}{3} \text{ ----------------------- (2)}$$

To find a_n:

w.k.t. $a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos\frac{n\pi x}{l}\, dx$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos\frac{n\pi x}{l}\, dx$$

Take $u = x^2$ $\quad v = \cos nx$

$u' = 2x$ $\quad v_1 = \frac{\sin nx}{n}$

$u'' = 2$ $\quad v_2 = -\frac{\cos nx}{n^2}$

$u''' = 0$ $\quad v_3 = -\frac{\sin nx}{n^3}$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left\{\left[\pi^2\left(\frac{\sin n\pi}{n}\right) - 2\pi\left(-\frac{\cos n\pi}{n^2}\right) + 2\left(-\frac{\sin n\pi}{n^3}\right)\right] - \left[(0^2)\left(-\frac{\sin 0}{n}\right) - 2(0)\left(-\frac{\cos 0}{n^2}\right) + 2\left(-\frac{\sin 0}{n^3}\right)\right]\right\}$$

$$= \frac{2}{\pi}\left\{\left(-\frac{2\pi(-1)^n}{n^2}\right)\right\}$$

$$\boldsymbol{a_n = -\frac{4(-1)^n}{n^2}} \text{ ----------------------- } \mathbf{(3)}$$

Substituting (2) & (3) in (1), we have

$$f(x) = -\frac{2\pi^2}{3} + \sum_{n=1}^{\infty}\left(-\frac{4(-1)^n}{n^2}\right)\cos nx$$

$$\boldsymbol{f(x) = \frac{2\pi^2}{3} + 4\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n^2}\right) cos\ nx} \text{ ------------------- } \textbf{proof (i)}$$

(ii) w.k.t. the Fourier half-range sine Series expansion is

$$\boldsymbol{f(x) = \sum_{n=1}^{\infty} b_n \sin nx} \text{ --------------------- } \mathbf{(4)}$$

To find b_n:

w.k.t. $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx\, dx$$

Take $u = x^2$ $\quad v = \sin nx$

$u' = 2x$ $\quad v_1 = -\frac{\cos nx}{n}$

$u'' = 2$ $\quad v_2 = -\frac{\sin nx}{n^2}$

$u''' = 0$ $\quad v_3 = \frac{\cos nx}{n^3}$

$$= \frac{2}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - 2x\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left\{\left[\pi^2\left(-\frac{\cos n\pi}{n}\right) - 2\pi\left(-\frac{\sin n\pi}{n^2}\right) + 2\left(\frac{\cos n\pi}{n^3}\right)\right] - \left[(0^2)\left(-\frac{\cos 0}{n}\right) - 2(0)\left(-\frac{\sin 0}{n^2}\right) + 2\left(\frac{\cos 0}{n^3}\right)\right]\right\}$$

$$= \frac{2}{\pi}\left\{\left[\pi^2\left(-\frac{(-1)^n}{n}\right) + 2\left(\frac{(-1)^n}{n^3}\right)\right] - \left[2\left(\frac{1}{n^3}\right)\right]\right\}$$

$$= \frac{2}{\pi}\left\{\pi^2\left(\frac{(-1)^{n+1}}{n}\right) + \frac{2}{n^3}[(-1)^n - 1]\right\}$$

$$b_n = \begin{cases} \frac{2}{\pi}\left[\frac{\pi^2}{n} - \frac{4}{n^3}\right] & \text{if } n \text{ is odd} \\ \frac{2}{\pi}\left[-\frac{\pi^2}{n}\right] & \text{if } n \text{ is even} \end{cases} \quad \text{----------------------- (5)}$$

Substituting (5) in (4), we have

$$f(x) = \sum_{n=1,3,5,\dots}^{\infty}\left(\frac{2}{\pi}\left[\frac{\pi^2}{n} - \frac{4}{n^3}\right]\right)\sin nx + \sum_{n=2,4,6,\dots}^{\infty}\left(\frac{2}{\pi}\left[-\frac{\pi^2}{n}\right]\right)\sin nx$$

$$f(x) = \frac{2}{\pi}\sum_{n=1,3,5,\dots}^{\infty}\left(\frac{\pi^2}{n} - \frac{4}{n^3}\right)\sin nx - 2\pi\sum_{n=2,4,6,\dots}^{\infty}\left(\frac{1}{n}\right)\sin nx \text{ --------- proof (ii)}$$

3. Find (i) the Fourier half-range cosine series and (ii) the Fourier half-range sine series of $f(x) = \begin{cases} x & in\ 0 < x < 1 \\ 2 - x & in\ 1 < x < 2 \end{cases}$.

Solution:

Given: $f(x) = \begin{cases} x & in\ 0 < x < 1 \\ 2 - x & in\ 1 < x < 2 \end{cases}$ in $(0, 2)$

(i) w.k.t. the Fourier half-range cosine Series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} \text{ --------------------- (1)}$$

To find a_0:

w.k.t. $a_0 = \frac{2}{2}\int_0^2 f(x)\,dx$

$$= \int_0^1 x\,dx + \int_1^2 (2 - x)\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2$$

$$= \frac{1}{2} + 4 - \frac{4}{2} - 2 + \frac{1}{2}$$

$$a_0 = 1 \text{ ----------------------- (2)}$$

To find a_n:

w.k.t. $a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx$

$$= \int_0^1 x\cos\frac{n\pi x}{2}\,dx + \int_1^2 (2 - x)\cos\frac{n\pi x}{2}\,dx$$

Take $u = x$ $\qquad\qquad v = \cos\frac{n\pi x}{2}$

$u' = 1 \qquad\qquad v_1 = \frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}$

$u'' = 0 \qquad\qquad v_2 = -\frac{\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}$

$$= \left[x\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - 1\left(-\frac{\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right]_0^1 + \left[(2-x)\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) + 1\left(-\frac{\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right]_1^2$$

$$= 1\left(\frac{\sin\frac{n\pi}{2}}{\frac{n\pi}{2}}\right) - 1\left(-\frac{\cos\frac{n\pi}{2}}{\frac{n^2\pi^2}{4}}\right) - 0\left(\frac{\sin 0}{\frac{n\pi}{2}}\right) + 1\left(-\frac{\cos 0}{\frac{n^2\pi^2}{4}}\right) + (2-2)\left(\frac{\sin n\pi}{\frac{n\pi}{2}}\right) +$$
$$1\left(-\frac{\cos n\pi}{\frac{n^2\pi^2}{4}}\right) - (2-1)\left(\frac{\sin\frac{n\pi}{2}}{\frac{n\pi}{2}}\right) - 1\left(-\frac{\cos\frac{n\pi}{2}}{\frac{n^2\pi^2}{4}}\right)$$

$$= \frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2} - \frac{4}{n^2\pi^2}\cos n\pi - \frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2}$$

$$= \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2}[1 + (-1)^n]$$

$$= \begin{cases} 0 & if\ n\ is\ odd \\ \frac{2}{m^2\pi^2}[(-1)^m - 1] & if\ n\ is\ even\ and = 2m \end{cases}$$

$$= \begin{cases} 0 & if\ m = \frac{n}{2}\ is\ even \\ -\frac{4}{m^2\pi^2} & if\ m = \frac{n}{2}\ is\ odd \end{cases}$$

$$\boldsymbol{a_n} = \begin{cases} 0 & if\ n\ is\ a\ multiple\ of\ 4 \\ -\frac{16}{n^2\pi^2} & if\ \ n\ is\ even, but\ not\ multiple\ of\ 4 \end{cases} \qquad \text{------------------------ (3)}$$

Substituting (2) & (3) in (1), we have

$$f(x) = \frac{1}{2} + \sum_{n=2,4,6,\ldots}^{\infty}\left(-\frac{16}{n^2\pi^2}\right)\cos\frac{n\pi x}{2}$$

$$f(x) = \frac{1}{2} - \frac{16}{\pi^2}\sum_{n=2,4,6,\ldots}^{\infty}\left(\frac{1}{n^2}\right) cos\frac{n\pi x}{2}$$

$$\boldsymbol{f(x) = \frac{1}{2} - \frac{16}{\pi^2}\left[\frac{cos\pi x}{2^2} + \frac{cos3\pi x}{6^2} + \frac{cos5\pi x}{10^2} + \cdots\right]} \text{ ------------------- } \textbf{proof (i)}$$

(ii) w.k.t. the Fourier half-range sine Series expansion is

$$\boldsymbol{f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}} \text{ --------------------- } \textbf{(4)}$$

To find b_n:

w.k.t. $b_n = \frac{2}{2}\int_0^2 f(x) \sin\frac{n\pi x}{2} dx$

$$= \int_0^1 x \sin\frac{n\pi x}{2} dx + \int_1^2 (2-x) \sin\frac{n\pi x}{2} dx$$

Take $u = x$ $\qquad v = \sin\frac{n\pi x}{2}$

$u' = 1$ $\qquad v_1 = -\frac{\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}$

$u'' = 0$ $\qquad v_2 = -\frac{\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}$

$$= \left[x\left(-\frac{\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - 1\left(-\frac{\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right]_0^1 + \left[(2-x)\left(-\frac{\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) + 1\left(-\frac{\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right]_1^2$$

$$= 1\left(-\frac{\cos\frac{n\pi}{2}}{\frac{n\pi}{2}}\right) - 1\left(-\frac{\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{4}}\right) - 0\left(-\frac{\cos 0}{\frac{n\pi}{2}}\right) + 1\left(-\frac{\sin 0}{\frac{n^2\pi^2}{4}}\right) +$$

$$(2-2)\left(-\frac{\cos n\pi}{\frac{n\pi}{2}}\right) + 1\left(-\frac{\sin n\pi}{\frac{n^2\pi^2}{4}}\right) - (2-1)\left(-\frac{\cos\frac{n\pi}{2}}{\frac{n\pi}{2}}\right) - 1\left(-\frac{\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{4}}\right)$$

$$= \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$= \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\boldsymbol{b_n = \begin{cases} \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2} & if\ n\ is\ odd \\ 0 & if\ n\ is\ even \end{cases}}$$ **------------------------ (5)**

Substituting (5) in (4), we have

$$f(x) = \sum_{n=1,3,5,\dots}^{\infty} \left(\frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\sin\frac{n\pi x}{2}$$

$$f(x) = \frac{8}{\pi^2}\sum_{n=1,3,5,\dots}^{\infty} \left(\frac{1}{n^2} sin\frac{n\pi}{2}\right)\sin\frac{n\pi x}{2}$$

$$\boldsymbol{f(x) = \frac{8}{\pi^2}\left[\frac{1}{1^2} sin\frac{\pi x}{2} - \frac{1}{3^2} sin\frac{3\pi x}{2} + \frac{1}{5^2} sin\frac{5\pi x}{2} - \cdots\right]}$$ **------------------ proof (ii)**

4. Find the Fourier half-range of cosine series of the function $f(x) = (x+1)^2$ in $(-1, 0)$. Hence find the value of $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \infty$.

Solution:

Given: $f(x) = (x+1)^2$ in $(-1, 0)$

we take the function $f(x) = (-x+1)^2$ in $(0, 1)$

(i) w.k.t. the Fourier half-range cosine Series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x \quad \text{--------------------- (1)}$$

To find a_0:

w.k.t. $a_0 = \frac{2}{1}\int_0^1 f(x)\, dx$

$$= 2\int_0^1 (1-x)^2\, dx$$

$$= 2\left[\frac{(1-x)^3}{-3}\right]_0^1$$

$$= 2\left[\frac{1}{3}\right]$$

$$a_0 = \frac{2}{3} \quad \text{---------------------- (2)}$$

To find a_n:

w.k.t. $a_n = \frac{2}{1}\int_0^1 f(x) \cos n\pi x\, dx$

$$= 2\int_0^1 (1-x) \cos n\pi x\, dx$$

Take $u = (1-x)^2$ $\quad\quad v = \cos n\pi x$

$u' = -2(1-x)$ $\quad\quad v_1 = \frac{\sin n\pi x}{n\pi}$

$u'' = -2(-1) = 2$ $\quad\quad v_2 = -\frac{\cos n\pi x}{n^2\pi^2}$

$u''' = 0$ $\quad\quad v_3 = -\frac{\sin n\pi x}{n^3\pi^3}$

$$= 2\left[(1-x)^2\left(\frac{\sin n\pi x}{n\pi}\right) + 2(1-x)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + 2\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= 2\left[(1-1)^2\left(\frac{\sin n\pi}{n\pi}\right) + 2(1-1)\left(-\frac{\cos n\pi}{n^2\pi^2}\right) + 2\left(-\frac{\sin n\pi}{n^3\pi^3}\right)\right] -$$

$$\left[(1-0)^2\left(\frac{\sin 0}{n\pi}\right) + 2(1-0)\left(-\frac{\cos 0}{n^2\pi^2}\right) + 2\left(-\frac{\sin 0}{n^3\pi^3}\right)\right]$$

$$= 2\left[\frac{2}{n^2\pi^2}\right]$$

$$\boldsymbol{a_n = \frac{4}{n^2\pi^2}} \text{ ----------------------- } \mathbf{(3)}$$

Substituting (2) & (3) in (1), we have

$$f(x) = \frac{\frac{2}{3}}{2} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2\pi^2}\right)\cos n\pi x$$

$$f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)\cos n\pi x$$

Put $x = 0$ is a point of continuity for $f(x)$

$$(1-x)^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)\cos n\pi x$$

$$\frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)\cos 0 = (1-0)^2$$

$$\frac{4}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right) = 1 - \frac{1}{3}$$

$$\frac{4}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\infty\right] = \frac{2}{3}$$

$$\boldsymbol{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\infty = \frac{\pi^2}{6}}$$

5. Find the half-range sine series of the function $\boldsymbol{f(x) = \pi - x}$ in $\boldsymbol{(\pi, 2\pi)}$ by suitably extending $\boldsymbol{f(x)}$ in $\boldsymbol{(0, \pi)}$. Deduce the sum of the series $\boldsymbol{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\infty}$.

Solution:

Given: $f(x) = \pi - x$ in $(0, \pi)$

w.k.t. the Fourier half-range sine Series expansion is

$$\boldsymbol{f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}} \text{ --------------------- } \mathbf{(1)}$$

To find $\boldsymbol{b_n}$:

w.k.t. $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$

$$b_n = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\sin nx\, dx$$

Take $u = \pi - x$ $\quad\quad\quad\quad$ $v = \sin nx$

$u' = -1$ $\quad\quad\quad\quad$ $v_1 = -\frac{\cos nx}{n}$

$u'' = 0$ $\quad\quad\quad\quad$ $v_2 = -\frac{\sin nx}{n^2}$

$$= \frac{2}{\pi}\left[(\pi - x)\left(-\frac{\cos nx}{n}\right) + 1\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[(\pi - \pi)\left(-\frac{\cos n\pi}{n}\right) + 1\left(-\frac{\sin n\pi}{n^2}\right)\right] - \left[(\pi - 0)\left(-\frac{\cos 0}{n}\right) + 1\left(-\frac{\sin 0}{n^2}\right)\right]$$

$$= \frac{2}{\pi}\left(\frac{\pi}{n}\right)$$

$$\boldsymbol{b_n = \frac{2}{n}} \text{ ----------------------- } \mathbf{(2)}$$

Substitute (2) in (1), we have

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{2}{n}\right)\sin nx$$

Put $x = \frac{\pi}{2}$ is a point of continuity for $f(x)$

$$2\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)\sin nx = \pi - \frac{\pi}{2}$$

$$\frac{\sin\frac{\pi}{2}}{1} + \frac{\sin\frac{3\pi}{2}}{3} + \frac{\sin\frac{5\pi}{2}}{5} + \frac{\sin\frac{7\pi}{2}}{7} + \cdots + \infty = \frac{\pi}{4}$$

$$\boldsymbol{\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \infty = \frac{\pi}{4}}$$

6. Find the half-range sine series of $f(x)$ in (0, λ) given that $f(x) = \begin{cases}(\lambda - c)x & in\ (0, c)\\ (\lambda - x)c & in\ (c, \lambda)\end{cases}$.

Solution:

Given: $f(x) = \begin{cases}(\lambda - c)x & in\ (0, c)\\ (\lambda - x)c & in\ (c, \lambda)\end{cases}$ in $(0, \lambda)$

w.k.t. the Fourier half-range sine Series expansion is

$$\boldsymbol{f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\lambda}} \text{ --------------------- } \mathbf{(1)}$$

To find b_n:

w.k.t. $b_n = \frac{2}{\lambda}\int_0^{\lambda} f(x)\sin\frac{n\pi x}{\lambda}dx$

$$b_n = \frac{2}{\lambda}\int_0^c f(x)\sin\frac{n\pi x}{\lambda}dx + \frac{2}{\lambda}\int_c^\lambda f(x)\sin\frac{n\pi x}{\lambda}dx$$

$$= \frac{2}{\lambda}\left\{\int_0^c(\lambda - c)x\sin\frac{n\pi x}{\lambda}dx + \int_c^\lambda(\lambda - x)c\sin\frac{n\pi x}{\lambda}dx\right\}$$

$$= \frac{2}{\lambda}\left\{\left[(\lambda - c)\int_0^c x\sin\frac{n\pi x}{\lambda}dx\right] + \left[c\int_c^\lambda(\lambda - x)\sin\frac{n\pi x}{\lambda}dx\right]\right\}$$

Take $u = x$ $\qquad v = \sin\frac{n\pi x}{\lambda}$ $\qquad u = \lambda - x$

$u' = 1$ $\qquad v_1 = -\frac{\cos\frac{n\pi x}{\lambda}}{\frac{n\pi}{\lambda}}$ $\qquad u' = -1$

$u'' = 0$ $\qquad v_2 = -\frac{\sin\frac{n\pi x}{\lambda}}{\frac{n^2\pi^2}{\lambda^2}}$ $\qquad u'' = 0$

$$= \frac{2(\lambda-c)}{\lambda}\left[x\left(-\frac{\cos\frac{n\pi x}{\lambda}}{\frac{n\pi}{\lambda}}\right) - 1\left(-\frac{\sin\frac{n\pi x}{\lambda}}{\frac{n^2\pi^2}{\lambda^2}}\right)\right]_0^c + \frac{2c}{\lambda}\left[(\lambda - x)\left(-\frac{\cos\frac{n\pi x}{\lambda}}{\frac{n\pi}{\lambda}}\right) + 1\left(-\frac{\sin\frac{n\pi x}{\lambda}}{\frac{n^2\pi^2}{\lambda^2}}\right)\right]_c^\lambda$$

$$= \frac{2(\lambda-c)}{\lambda}\left[-x\left(\frac{\cos\frac{n\pi x}{\lambda}}{\frac{n\pi}{\lambda}}\right) + 1\left(\frac{\sin\frac{n\pi x}{\lambda}}{\frac{n^2\pi^2}{\lambda^2}}\right)\right]_0^c + \frac{2c}{\lambda}\left[-(\lambda - x)\left(\frac{\cos\frac{n\pi x}{\lambda}}{\frac{n\pi}{\lambda}}\right) - 1\left(\frac{\sin\frac{n\pi x}{\lambda}}{\frac{n^2\pi^2}{\lambda^2}}\right)\right]_c^\lambda$$

$$= \frac{2(\lambda-c)}{\lambda}\left[-c\left(\frac{\cos\frac{n\pi c}{\lambda}}{\frac{n\pi}{\lambda}}\right) + 1\left(\frac{\sin\frac{n\pi c}{\lambda}}{\frac{n^2\pi^2}{\lambda^2}}\right) + 0\left(\frac{\cos 0}{\frac{n\pi}{\lambda}}\right) - 1\left(\frac{\sin 0}{\frac{n^2\pi^2}{\lambda^2}}\right)\right] +$$

$$\frac{2c}{\lambda}\left[-(\lambda - \lambda)\left(\frac{\cos n\pi}{\frac{n\pi}{\lambda}}\right) - 1\left(\frac{\sin n\pi}{\frac{n^2\pi^2}{\lambda^2}}\right) + (\lambda - c)\left(\frac{\cos\frac{n\pi c}{\lambda}}{\frac{n\pi}{\lambda}}\right) + 1\left(\frac{\sin\frac{n\pi c}{\lambda}}{\frac{n^2\pi^2}{\lambda^2}}\right)\right]$$

$$= \frac{2(\lambda-c)}{\lambda}\left[-c\left(\frac{\cos\frac{n\pi c}{\lambda}}{\frac{n\pi}{\lambda}}\right) + 1\left(\frac{\sin\frac{n\pi c}{\lambda}}{\frac{n^2\pi^2}{\lambda^2}}\right)\right] + \frac{2c}{\lambda}\left[(\lambda - c)\left(\frac{\cos\frac{n\pi c}{\lambda}}{\frac{n\pi}{\lambda}}\right) + 1\left(\frac{\sin\frac{n\pi c}{\lambda}}{\frac{n^2\pi^2}{\lambda^2}}\right)\right]$$

$$= -\frac{2c(\lambda-c)}{\lambda}\left(\frac{\cos\frac{n\pi c}{\lambda}}{\frac{n\pi}{\lambda}}\right) + \frac{2(\lambda-c)}{\lambda}\left(\frac{\sin\frac{n\pi c}{\lambda}}{\frac{n^2\pi^2}{\lambda^2}}\right) + \frac{2c(\lambda-c)}{\lambda}\left(\frac{\cos\frac{n\pi c}{\lambda}}{\frac{n\pi}{\lambda}}\right) + \frac{2c}{\lambda}\left(\frac{\sin\frac{n\pi c}{\lambda}}{\frac{n^2\pi^2}{\lambda^2}}\right)$$

$$= \frac{2(\lambda-c)}{\lambda}\left(\frac{\sin\frac{n\pi c}{\lambda}}{\frac{n^2\pi^2}{\lambda^2}}\right) + \frac{2c}{\lambda}\left(\frac{\sin\frac{n\pi c}{\lambda}}{\frac{n^2\pi^2}{\lambda^2}}\right)$$

$$\boldsymbol{b_n = \frac{2\lambda^2}{n^2\pi^2}\sin\frac{n\pi c}{\lambda}} \quad \textbf{------------------------ (2)}$$

Substitute (2) in (1), we have

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{2\lambda^2}{n^2\pi^2} sin \frac{n\pi c}{\lambda} \right) \sin \frac{n\pi x}{\lambda}$$

$$\boldsymbol{f(x) = \frac{2\lambda^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} sin \frac{n\pi c}{\lambda} \sin \frac{n\pi x}{\lambda}}$$

7. Find the half-range cosine series of $f(x) = sin\, x$ in $(0, \pi)$.

Solution:

Given: $f(x) = \sin x$ in $(0, \pi)$

w.k.t. the Fourier half-range cosine Series expansion is

$$\boldsymbol{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \ cos\ nx} \text{ --------------------- (1)}$$

To find a_0:

w.k.t. $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx$

$$= \frac{2}{\pi} \int_0^{\pi} \sin x\, dx$$

$$= \frac{2}{\pi} [-\cos x]_0^{\pi}$$

$$= \frac{2}{\pi} [-\cos \pi + \cos 0]$$

$$= \frac{2}{\pi} [-(-1) + 1]$$

$$\boldsymbol{a_0 = \frac{4}{\pi}} \text{ ----------------------- (2)}$$

To find a_n:

w.k.t. $a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\, dx$

$$= \frac{2}{\pi} \int_0^{\pi} \sin x \cos nx\, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \left[\frac{\sin(n+1)x - \sin(n-1)x}{2} \right] dx$$

$$= \frac{1}{\pi} \left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1} \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{\cos 0}{n+1} - \frac{\cos 0}{n-1} \right]$$

$$= \frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{1}{n+1} - \frac{1}{n-1}\right)(1 - (-1)^{n-1}\right]$$

$$= -\frac{2}{\pi(n^2-1)}[1 - (-1)^{n-1}]$$

$$\boldsymbol{a_n} = \begin{cases} 0 & if\ n\ is\ odd \\ -\frac{4}{\pi(n^2-1)} & if\ n\ is\ even \end{cases} \quad \textbf{-----------------------\ (3)}$$

To find $\boldsymbol{a_1}$:

w.k.t. $a_1 = \frac{2}{\pi}\int_0^{\pi} f(x)\cos x\, dx$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x \cos x\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{\sin 2x}{2} dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin 2x\, dx$$

$$= \frac{1}{\pi}\left[-\frac{cos2x}{2}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}[-\cos 2\pi + \cos 0]$$

$$= \frac{1}{2\pi}[-1 + 1]$$

$$\boldsymbol{a_1 = 0} \quad \textbf{----------------------\ (4)}$$

Substitute (2), (3), (4) in (1) we have

$$f(x) = \frac{\frac{4}{\pi}}{2} + 0 + \sum_{n=2,4,6,\dots}^{\infty}\left(-\frac{4}{\pi(n^2-1)}\right)\cos nx$$

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=2,4,6,\dots}^{\infty}\left(\frac{1}{n^2-1}\right)\cos nx$$

$$\boldsymbol{\sin x = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{4n^2-1}\right)\cos 2nx}$$

8. **Find the half-range cosine series of $f(x) = x\, sin\, x$ in $(0, \pi)$. Deduce the sum of the series $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \cdots \infty$.**

Solution:

Given: $f(x) = x\, sin\, x$ in $(0, 2\pi)$

w.k.t. the Fourier Series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \text{ --------------------- (1)}$$

To find a_0:

$$\text{w.k.t. } a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi} x \sin x\, dx\right\}$$

$$= \frac{2}{\pi}[x(-\cos x) - 1(-\sin x)]_0^{\pi}$$

$$= \frac{2}{\pi}[(-\pi\cos\pi + \sin\pi) - (0 - \sin 0)]$$

$$= \frac{2}{\pi}[\pi]$$

$$a_0 = 2 \text{ ---------------------- (2)}$$

To find a_n:

$$\text{w.k.t. } a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi} x\ \sin x \cos nx\, dx\right\}$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi} x\left[\frac{sin(1+n)x + \sin(1-n)x}{2}\right]dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} x\, sin(1+n)x\ dx + \int_0^{\pi} x\, sin(1-n)x\ dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{x(-\cos(1+n)x)}{1+n} + \frac{\sin(1+n)x}{(1+n)^2}\right]_0^{\pi} + \left[\frac{x\,(-\cos(1-n)x}{1-n} + \frac{\sin(1-n)x}{(1-n)^2}\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\begin{matrix}\left[\frac{-\pi\cos(1+n)\pi}{1+n} + \frac{\sin(1+n)\pi}{(1+n)^2} - \frac{0}{n+1} - \frac{0}{(n+1)^2}\right] + \\ \left[\frac{-\pi\cos(1-n)\pi}{1-n} + \frac{\sin(1-n)\pi}{(1-n)^2} - \frac{0}{n-1} - \frac{0}{(n-1)^2}\right]\end{matrix}\right\}$$

$= \frac{1}{\pi}\left[\frac{-\pi\cos(1+n)\pi}{1+n} - \frac{-\pi\cos(1-n)\pi}{1-n}\right]$

$= \frac{\pi}{\pi}\left[\frac{\cos\pi\cos n\pi - \sin\pi\sin n\pi}{1+n} + \frac{\cos\pi\cos n\pi + \sin\pi\sin n\pi}{1-n}\right]$

$= \frac{\cos n\pi}{1+n} + \frac{\cos n\pi}{1-n}$

$= (-1)^n\left[\frac{1}{1+n} - \frac{1}{1-n}\right]$

$= \frac{2(-1)^n}{1-n^2}$

$\boldsymbol{a_n = \frac{2(-1)^{n-1}}{n^2-1}}$ **------------------------ (3)**

To find a_1:

w.k.t. $a_1 = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\, dx$

$= \frac{2}{\pi}\left\{\int_0^\pi x\ \sin x\cos x\, dx\right\}$

$= \frac{2}{\pi}\left\{\int_0^\pi x\left(\frac{\sin 2x}{2}\right)dx\right\}$

$= \frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - \left(-\frac{\sin 2x}{4}\right)\right]_0^\pi$

$= \frac{1}{\pi}\left[\pi\left(-\frac{\cos 2\pi}{2}\right) - \left(-\frac{\sin 2\pi}{4}\right) - 0\left(-\frac{\cos 0}{2}\right) - \left(-\frac{\sin 0}{4}\right)\right]$

$= \frac{1}{\pi}\left[\pi\left(-\frac{\cos 2\pi}{2}\right)\right]$

$\boldsymbol{a_1 = -\frac{1}{2}}$ **----------------------------- (4)**

Substituting (2), (3), (4), in (1), we have

$f(x) = \frac{2}{2} - \frac{1}{2}\cos x - \sum_{n=2}^\infty\left(\frac{\mathbf{2(-1)^{n-1}}}{\mathbf{n^2-1}}\right)\cos nx$

$\boldsymbol{f(x) = 1 - \frac{1}{2}\cos x - 2\sum_{n=2}^\infty\left(\frac{(-1)^{n-1}}{(n-1)(n+1)}\right)\cos nx}$

Put $x = \frac{\pi}{2}$ is a point of continuity of $f(x)$, we have

$\frac{\pi}{2}sin\frac{\pi}{2} = 1 - \frac{1}{2}\cos\frac{\pi}{2} + 2\sum_{n=2}^\infty\left(\frac{(-1)^n}{(n-1)(n+1)}\right)\cos\frac{n\pi}{2}$

$\frac{\pi}{2} = 1 + 2\left[-\frac{1}{1.3}\cos\pi - \frac{1}{3.5}\cos 2\pi - \frac{1}{5.7}\cos 3\pi - \cdots\infty\right]$

$2\left[-\frac{1}{1.3}(-1) - \frac{1}{3.5}(1) - \frac{1}{5.7}(-1) - \cdots\infty\right] = \frac{\pi}{2} - 1$

$$2\left[\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-\cdots\infty\right]=\frac{\pi-2}{2}$$

$$\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-\cdots\infty=\frac{\pi-2}{4}$$

TRY YOURSELF

1. Find the half-range sine series of $f(x)=\sin ax$ in$(0,l)$.
2. Find the half-range sine series of $f(x)=\frac{\sinh ax}{\sinh a\pi}$ in $(0,\pi)$.

PARSEVALS IDENTITY THEOREM

- Fourier series in $(0,2l)$

$$\frac{1}{2l}\int_0^l[f(x)]^2dx=\frac{a_0^2}{4}+\frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

- Fourier half-range cosine series in $(0,l)$

$$\frac{2}{l}\int_0^l[f(x)]^2dx=\frac{a_0^2}{2}+\sum_{n=1}^{\infty}a_n^2$$

- Fourier half-range cosine series in $(0,\pi)$

$$\frac{2}{\pi}\int_0^\pi[f(x)]^2dx=\frac{a_0^2}{2}+\sum_{n=1}^{\infty}a_n^2$$

- Fourier half-range sine series in $(0,l)$

$$\frac{2}{l}\int_0^l[f(x)]^2dx=\sum_{n=1}^{\infty}b_n^2$$

- Fourier half-range sine series in $(0,\pi)$

$$\frac{2}{\pi}\int_0^\pi[f(x)]^2dx=\sum_{n=1}^{\infty}b_n^2$$

1. Find the half-range cosine series of $f(x)=x$ in $(0,1)$. Deduce the sum of the series $\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots\infty$.

Solution:

Given: $f(x)=x$ in $(0,1)$

w.k.t. the Fourier half-range cosine Series expansion is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos n\pi x \text{ --------------------- (1)}$$

To find a_0:

w.k.t. $a_0=\frac{2}{1}\int_0^1 f(x)\,dx$

$$=2\int_0^1 x\,dx$$

$$=2\left[\frac{x^2}{2}\right]_0^1$$

$$= 2\left[\frac{1}{2}\right]$$

$\boldsymbol{a_0 = 1}$ **----------------------- (2)**

To find $\boldsymbol{a_n}$:

w.k.t. $a_n = \frac{2}{1}\int_0^1 f(x)\cos n\pi x\, dx$

$$= 2\int_0^1 x\cos n\pi x\, dx$$

Take $u = x$ $\qquad v = \cos n\pi x$

$u' = 1 \qquad v_1 = \frac{\sin n\pi x}{n\pi}$

$u'' = 0 \qquad v_2 = -\frac{\cos n\pi x}{n^2\pi^2}$

$$= 2\left[x\left(\frac{\sin n\pi x}{n\pi}\right) - 1\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$= 2\left[1\left(\frac{\sin n\pi}{n\pi}\right) - 1\left(-\frac{\cos n\pi}{n^2\pi^2}\right) - 0\left(\frac{\sin 0}{n\pi}\right) + 1\left(-\frac{\cos 0}{n^2\pi^2}\right)\right]$$

$$= 2\left[\frac{(-1)^n}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right]$$

$$= \frac{2}{n^2\pi^2}[(-1)^n - 1]$$

$$\boldsymbol{a_n = \begin{cases} -\frac{4}{n^2\pi^2} & if\ n\ is\ odd \\ 0 & if\ n\ is\ even \end{cases}} \text{ ----------------------- } \mathbf{(3)}$$

Substituting (2) & (3) in (1), we have

$$f(x) = \frac{1}{2} + \sum_{n=1,3,5,\ldots}^{\infty}\left(-\frac{4}{n^2\pi^2}\right)\cos n\pi x$$

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\left(\frac{1}{n^2}\right)\cos n\pi x$$

Now, we apply the Parsevals identity theorem

$$\frac{2}{1}\int_0^1 [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$2\int_0^1 x^2 dx = \frac{1}{2} + \sum_{n=1,3,5,\dots}^{\infty} \left(-\frac{4}{n^2\pi^2}\right)^2$

$2\left[\frac{x^3}{3}\right]_0^1 = \frac{1}{2} + \sum_{n=1,3,5,\dots}^{\infty} \frac{16}{n^4\pi^4}$

$\frac{16}{\pi^4}\sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^4} = \frac{2}{3} - \frac{1}{2}$

$\frac{16}{\pi^4}\sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^4} = \frac{1}{6}$

$\boldsymbol{\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots \infty = \frac{\pi^4}{96}}$

2. **Find the half-range cosine series of $f(x) = (\pi - x)^2$ in $(0, \pi)$. Hence find the sum of the series $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots \infty$.**

Solution:

Given: $f(x) = (\pi - x)^2$ in $(0, \pi)$

w.k.t. the Fourier half-range cosine Series expansion is

$$\boldsymbol{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx} \text{ --------------------- (1)}$$

To find a_0:

w.k.t. $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$

$= \frac{2}{\pi}\int_0^{\pi} (\pi - x)^2\, dx$

$= \frac{2}{\pi}\left[\frac{(\pi - x)^3}{-3}\right]_0^{\pi}$

$= \frac{2}{\pi}\left[\frac{\pi^3}{3}\right]$

$$\boldsymbol{a_0 = \frac{2\pi^2}{3}} \text{ ----------------------- (2)}$$

To find a_n:

w.k.t. $a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\, dx$

$= \frac{2}{\pi}\int_0^{\pi} (\pi - x)^2 \cos nx\, dx$

Take $u = (\pi - x)^2$ $\qquad\qquad v = \cos nx$

$u' = -2(\pi - x)$ $\qquad$ $v_1 = \frac{\sin nx}{n}$

$u'' = 2$ $\qquad$ $v_2 = -\frac{\cos nx}{n^2}$

$u''' = 0$ $\qquad$ $v_3 = -\frac{\sin nx}{n^3}$

$$= \frac{2}{\pi}\left[(\pi - x)^2\left(\frac{\sin nx}{n}\right) + 2(\pi - x)\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[(\pi - \pi)^2\left(\frac{\sin n\pi}{n}\right) + 2(\pi - \pi)\left(-\frac{\cos n\pi}{n^2}\right) + 2\left(-\frac{\sin n\pi}{n^3}\right) - (\pi - 0)^2\left(\frac{\sin 0}{n}\right) - 2(\pi - 0)\left(-\frac{\cos 0}{n^2}\right) + 2\left(-\frac{\sin 0}{n^3}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{2\pi}{n^2}\right]$$

$$\boldsymbol{a_n = \frac{4}{n^2}} \text{------------------------ } \mathbf{(3)}$$

Substituting (2) & (3) in (1), we have

$$f(x) = \frac{\frac{2\pi^2}{3}}{2} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\right)\cos nx$$

$$f(x) = \frac{2\pi^2}{6} + 4\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)\cos nx$$

Now, we apply the Parsevals identity theorem

$$\frac{2}{\pi}\int_0^\pi [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$$\frac{2}{\pi}\int_0^\pi (\pi - x)^4 dx = \frac{\frac{4\pi^4}{9}}{2} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\right)^2$$

$$\frac{2}{\pi}\left[\frac{(\pi - x)^5}{-5}\right]_0^\pi = \frac{2\pi^4}{9} + 16\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{2}{\pi}\left[\frac{\pi^5}{5}\right] = \frac{2\pi^4}{9} + 16\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$16\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{2\pi^4}{5} - \frac{2\pi^4}{9}$$

$$16\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{8\pi^4}{45}$$

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots\infty=\frac{\pi^4}{90}$$

TRY YOURSELF

1. Find the half-range of sine series of $f(x)=\begin{cases} x & in\ \left(0,\frac{\pi}{2}\right) \\ \pi-x & in\ \left(\frac{\pi}{2},\pi\right)\end{cases}$ in $(0,\pi)$. Hence find the sum of the series $\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots\infty$.
2. Find the half-range of cosine series of $f(x)=x(\pi-x)$ in $(0,\pi)$. Hence find the sum of the series $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots\infty$**.**

HARMONIC ANALYSIS

The process of finding the Fourier series for a function given by numerical values is known as harmonic analysis. In harmonic analysis the Fourier coefficient a_0, a_n and b_n of the function $y=f(x)$ in $(0,2\pi)$ are given by

a_0 = 2 [mean value of y in $(0,2\pi)$]

a_n = 2 [mean value of $y\cos nx$ in $(0,2\pi)$]

b_n = 2 [mean value of $y\sin nx$ in $(0,2\pi)$]

NOTE:

- The term $(a_1\cos x+b_1\sin x)$ is called the fundamental or first harmonic, the term $(a_2\cos 2x+b_2\sin 2x)$ is called the second harmonic and so on.
- The number of ordinates used should not be less than twice the number of highest harmonic to be found.

1. Obtain the first three harmonic in the Fourier series expansion in $(0,12)$ for the function $y=f(x)$ defined by the table given below:

x	0	1	2	3	4	5	6	7	8	9	10	11
y	1.8	1.1	0.3	0.16	0.5	1.5	2.16	1.88	1.25	1.30	1.76	2.00

Solution:

w.k.t. the first three harmonic in the Fourier series is

$$y=\frac{a_0}{2}+a_1\cos x+a_2\cos 2x+a_3\cos 3x+b_1\sin x+b_2\sin 2x+b_3\sin 3x \text{ ------- (1)}$$

Where the Fourier coefficient $a_0, a_1, a_2, a_3, b_1, b_2$ and b_3 are to be determined from the following table.

x	y	$\cos x$	$y\cos x$	$\cos 2x$	$y\cos 2x$	$\cos 3x$	$y\cos 3x$	$\sin x$	$y\sin x$	$\sin 2x$	$y\sin 2x$	$\sin 3x$	$y\sin 3x$
0	1.8	1	1	1	1.80	1.00	1.80	0.00	0.00	0.00	0.00	0.00	0.00
1	1.1	0.54	0.59	-0.42	-0.46	-0.99	-1.09	0.84	0.93	0.91	1.00	0.14	0.16
2	0.3	-0.42	-0.13	-0.65	-0.20	0.96	0.29	0.91	0.27	-0.76	-0.23	-0.28	-0.08
3	0.16	-0.99	-0.16	0.96	0.15	-0.91	-0.15	0.14	0.02	-0.28	-0.04	0.41	0.07
4	0.5	-0.65	-0.33	-0.15	-0.07	0.84	0.42	-0.76	-0.38	0.99	0.49	-0.54	-0.27
5	0.15	0.28	0.04	-0.84	-0.13	-0.76	-0.11	-0.96	-0.14	-0.54	-0.08	0.65	0.10
6	2.16	0.96	2.07	0.84	1.82	0.66	1.43	-0.28	-0.60	-0.54	-1.16	-0.75	-1.62
7	1.88	0.75	1.41	0.14	0.26	-0.55	-1.03	0.66	1.24	0.99	1.86	0.84	1.57
8	1.25	-0.15	-0.19	-0.96	-1.20	0.42	0.53	0.99	1.24	-0.29	-0.36	-0.91	-1.13
9	1.30	-0.91	-1.18	0.66	0.86	-0.29	-0.38	0.41	0.54	-0.75	-0.98	0.96	1.24
10	1.76	-0.84	-0.15	0.41	0.72	0.15	0.27	-0.54	-0.96	0.91	1.61	-0.99	-1.74
11	2.00	0.004	0.008	-0.99	-2.00	-0.01	-0.03	-1.00	-2.00	-0.01	-0.02	1.00	2.00
	14.36		2.978		1.56		1.95		0.15		2.10		0.29

$$a_0 = 2\left[\frac{14.36}{12}\right] = 2.39$$

$$a_1 = 2\left[\frac{2.978}{12}\right] = 0.49 \qquad b_1 = 2\left[\frac{0.15}{12}\right] = 0.03$$

$a_2 = 2\left[\frac{1.56}{12}\right] = 0.26$ $\qquad$ $b_2 = 2\left[\frac{2.10}{12}\right] = 0.35$

$a_3 = 2\left[\frac{1.95}{12}\right] = 0.33$ $\qquad$ $b_3 = 2\left[\frac{0.29}{12}\right] = 0.05$

$$y = \frac{2.39}{2} + (0.49\cos x + 0.03\sin x) + (0.26\cos 2x + 0.35\sin 2x) + (0.33\cos 3x + 0.05\sin 3x)$$

$$= \mathbf{1.19 + (0.49\cos x + 0.03\sin x) + (0.26\cos 2x + 0.35\sin 2x) + (0.33\cos 3x + 0.05\sin 3x)}$$

TRY YOURSELF

1. The following are 12 values of y corresponding to equidistant values of the angle x^0 in the range 0^0 to 360^0. Find the first three harmonics in the Fourier series expansion of y in $(0, 2\pi)$.

x^0	0	30	60	90	120	150	180	210	240	270	300	330
y	10.5	20.2	26.4	29.3	27.0	21.5	12.8	1.6	-11.2	-18.0	-15.8	-3.5

2. A function $y = f(x)$ is given by the following table of values. Make a harmonic analysis of the function in $(0, T)$ upto the second harmonic.

x	0	T/6	T/3	T/2	2T/3	5T/6	T
y	0	9.2	14.4	17.8	17.3	11.7	0

COMPLEX FORM OF FOURIER SERIES

- In $(-l, l)$ $f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{\frac{in\pi x}{l}}$ where $c_n = \frac{1}{2l}\int_{-l}^{l} f(x) e^{-\frac{in\pi x}{l}} dx$
- In $(0, 2l)$ $f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{\frac{in\pi x}{l}}$ where $c_n = \frac{1}{2l}\int_{0}^{2l} f(x) e^{-\frac{in\pi x}{l}} dx$
- In $(-\pi, \pi)$ $f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{inx}$ where $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx$
- In $(0, 2\pi)$ $f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{inx}$ where $c_n = \frac{1}{2\pi}\int_{0}^{2\pi} f(x) e^{-inx} dx$

1. Find the complex form of the Fourier series of $f(x) = e^x$ in $(0, 2)$.

Solution:

Given $f(x) = e^x$ in $(0, 2)$

w.k.t. the complex form of Fourier series is

$\boldsymbol{f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{in\pi x}}$ **----------------- (1)**

Where $\boldsymbol{c_n = \frac{1}{2}\int_0^2 f(x)\, e^{-in\pi x} dx}$

$$= \frac{1}{2}\int_0^2 e^x\, e^{-in\pi x} dx$$

$= \frac{1}{2}\int_0^2 \boldsymbol{e^{(1-in\pi)x}\,dx}$

$= \frac{1}{2}\left[\frac{e^{(1-in\pi)x}}{1-in\pi}\right]_0^2$

$= \frac{e^{2(1-in\pi)}}{2(1-in\pi)} - \frac{e^0}{2(1-in\pi)}$

$= \frac{e^{2(1-in\pi)}-1}{2(1-in\pi)}$

$= \frac{e^{2(1-in\pi)}-1}{2(1-in\pi)} \times \frac{1+in\pi}{1+in\pi}$

$= \frac{1+in\pi}{2(1+n^2\pi^2)}\left[e^2.e^{-i2n\pi} - 1\right]$

$= \frac{1+in\pi}{2(1+n^2\pi^2)}\left[e^2.(cos2n\pi - isin2n\pi) - 1\right]$ $\quad [\because \cos n\pi = (-1)^n$

$\boldsymbol{= \frac{1+in\pi}{2(1+n^2\pi^2)}\left[e^2 - 1\right]}$ **---------------- (2)** $\quad \sin n\pi = 0]$

Substituting (2) in (1), we have

$f(x) = \sum_{n=-\infty}^{\infty}\left(\frac{1+in\pi}{2(1+n^2\pi^2)}\left[e^2 - 1\right]\right) e^{in\pi x}$

$\boldsymbol{= \frac{e^2-1}{2}\sum_{n=-\infty}^{\infty}\left(\frac{1+in\pi}{1+n^2\pi^2}\right) e^{in\pi x}}$

2. Find the complex form of the Fourier series of $f(x) = e^{-ax}$ in $(-l, l)$.

Solution:

Given $f(x) = e^{-ax}$ in $(-l, l)$

w.k.t. the complex form of Fourier series is

$\boldsymbol{f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{\frac{in\pi x}{l}}}$ **----------------- (1)**

Where $\boldsymbol{c_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-\frac{in\pi x}{l}} dx}$

$= \frac{1}{2l}\int_{-l}^{l} e^{-ax}\, e^{-\frac{in\pi x}{l}} dx$

$= \frac{1}{2l}\int_{-l}^{l} \boldsymbol{e^{-(a+\frac{in\pi}{l})x}\, dx}$

$= \frac{1}{2l}\left[\frac{e^{-\left(a+\frac{in\pi}{l}\right)x}}{a+\frac{in\pi}{l}}\right]_{-l}^{l}$

$= \frac{1}{2l}\left[\frac{e^{-l\left(a+\frac{in\pi}{l}\right)}}{\left(a+\frac{in\pi}{l}\right)} - \frac{e^{l\left(a+\frac{in\pi}{l}\right)}}{\left(a+\frac{in\pi}{l}\right)}\right]$

$$= \frac{1}{2l}\left[\frac{e^{-l\left(\frac{al+in\pi}{l}\right)}}{\left(\frac{al+in\pi}{l}\right)} - \frac{e^{l\left(\frac{al+in\pi}{l}\right)}}{\left(\frac{al+in\pi}{l}\right)}\right]$$

$$= \frac{1}{2}\left[\frac{e^{-(al+in\pi)}}{al+in\pi} - \frac{e^{al+in\pi}}{al+in\pi}\right]$$

$$= \frac{1}{2(al+in\pi)}\left[e^{-al}.e^{-in\pi} - e^{al}.e^{in\pi}\right]$$

$$= \frac{1}{2(al+in\pi)}\left[e^{-al}.(-1)^n - e^{al}.(-1)^n\right] \qquad \left[\because e^{\pm in\pi} = (-1)^n\right]$$

$$= \frac{(-1)^n}{(al+in\pi)}\left[\frac{e^{al}-e^{-al}}{2}\right]$$

$$= \frac{(-1)^n}{(al+in\pi)}[\sinh al]$$

$$= \frac{(-1)^n \sinh al}{(al+in\pi)}\left[\frac{al-in\pi}{al-in\pi}\right]$$

$$= \frac{(-1)^n(al-in\pi)\sinh al}{a^2l^2+n^2\pi^2} \text{ ---------------- } (2)$$

Substituting (2) in (1), we have

$$f(x) = \sum_{n=-\infty}^{\infty}\left(\frac{(-1)^n(al-in\pi)\sinh al}{a^2l^2+n^2\pi^2}\right)e^{\frac{in\pi x}{l}}$$

$$= \sinh al \sum_{n=-\infty}^{\infty}\left(\frac{(-1)^n(al-in\pi)}{a^2l^2+n^2\pi^2}\right)e^{\frac{in\pi x}{l}}$$

3. **Find the complex form of the Fourier series of $f(x) = \sin x$ in $(0, \pi)$.**

Solution:

Given $f(x) = \sin x$ in $(0, \pi)$

w.k.t. the complex form of Fourier series is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i2nx} \text{ ----------------- } (1)$$

Where $c_n = \frac{1}{\pi}\int_0^2 f(x)\, e^{-in\pi x} dx$

$$= \frac{1}{\pi}\int_0^2 \sin x\, e^{-i2nx} dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-i2nx}}{1-4n^2}(-i2n\sin x - \cos x)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-i2n\pi}}{1-4n^2}(-i2n\sin\pi - \cos\pi) - \frac{e^0}{1-4n^2}(-i2n\sin 0 - \cos 0)\right]$$

$$= \frac{1}{\pi}\left[\frac{e^{-i2n\pi}}{1-4n^2}(1) + \frac{1}{1-4n^2}(1)\right]$$

$$= \frac{1}{\pi(1-4n^2)}\left[e^{-i2n\pi} + 1\right]$$

$$= -\frac{2}{\pi(4n^2-1)} \text{ ---------------- } (2)$$

Substituting (2) in (1), we have

$$f(x) = \sum_{n=-\infty}^{\infty}\left(= -\frac{2}{\pi(4n^2-1)}\right)e^{i2nx}$$

$$= -\frac{2}{\pi}\sum_{n=-\infty}^{\infty}\left(\frac{1}{4n^2-1}\right)e^{i2nx}$$

TRY YOURSELF

1. Find the complex form of the Fourier series of $f(x) = \cos ax$ in $(-\pi, \pi)$ where a is neither zero nor an integer.
2. Find the complex form of the Fourier series of $f(x) = \cos x$ in $(0, \pi)$.
3. Find the complex form of the Fourier series of $f(x) = e^{-x}$ in $(-\pi, \pi)$.
4. Find the complex form of the Fourier series of $f(x) = e^{ax}$ in $(0, 2l)$.

UNIT III

FOURIER TRANSFORMS

FOURIER INTEGRAL THEOREM

If $f(x)$ is a given function defined in $(-l, l)$ and satisfies the following conditions, then $(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos s(t-x)\, dt\, ds$. This is also known as Fourier integral formula.

CONDITION OF $f(x)$:

- $f(x)$ is well defined and single valued except at finite number of points in $(-l\,, l)$.
- $f(x)$ is periodic in $(-l, l)$.
- $f(x)$ and $f'(x)$ are piecewise continuous in $(-l, l)$.
- $\int_{-\infty}^\infty |f(x)|dx$ converges.

COMPLEX FORM OF FOURIER INTEGRAL

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-isx}\int_{-\infty}^\infty f(t)e^{ist}\, dt\, ds$$

FOURIER SINE INTEGRAL FORMULA

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin sx\, \left\{\int_0^\infty f(t)\sin st\ \, dt\right\} ds$$

FOURIER COSINE INTEGRAL FORMULA

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos sx\, \left\{\int_0^\infty f(t)\cos st\ \, dt\right\} ds$$

1. Find the Fourier integral of $f(x) = \begin{cases} 0 & , x < 0 \\ \frac{1}{2} & , x = 0 \\ e^{-x} & , x > 0 \end{cases}$. Verify the representation directly at the point $x = 0$.

Solution:

Given: $f(x) = \begin{cases} 0 & , x < 0 \\ \frac{1}{2} & , x = 0 \\ e^{-x} & , x > 0 \end{cases}$

w.k.t. the Fourier integral formula

$$\boldsymbol{f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos s(t-x)\,dt\,ds} \text{ ------------------ (1)}$$

$$= \frac{1}{\pi}\int_0^\infty \left\{\int_{-\infty}^0 f(t)\cos s(t-x)\,dt + \int_0^\infty f(t)\cos s(t-x)\,dt\right\} ds$$

Here $f(t) = \begin{cases} 0 & ,t<0 \\ \frac{1}{2} & ,t=0 \\ e^{-x} & ,t>0 \end{cases}$

$$= \frac{1}{\pi}\int_0^\infty \left\{\int_{-\infty}^0 0.\cos s(t-x)\,dt + \int_0^\infty e^{-t}\cos s(t-x)\,dt\right\} ds$$

$$= \frac{1}{\pi}\int_0^\infty \{\int_0^\infty e^{-t}\cos(st-sx)\,dt\} ds$$

$$\left\{\because \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]\right\}$$

$$= \frac{1}{\pi}\int_0^\infty \left[\frac{e^{-t}}{(-1)^2+s^2}[(-1)\cos(st-sx) + (s)\sin(st-sx)]\right]_0^\infty ds$$

$$= \boldsymbol{\frac{1}{\pi}\int_0^\infty \frac{\cos sx + s\sin sx}{s^2+1}\,ds} \text{ ----------------------- (2)}$$

Put $x = 0$ in (2), we have

$$f(0) = \frac{1}{\pi}\int_0^\infty \frac{\cos 0 + s\sin 0}{s^2+1}\,ds$$

$$= \frac{1}{\pi}\int_0^\infty \frac{1}{s^2+1}\,ds$$

$$= \frac{1}{\pi}[\tan^{-1} s]_0^\infty$$

$$= \frac{1}{\pi}[\tan^{-1}\infty - \tan^{-1} 0]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2}\right]$$

$$\boldsymbol{f(0) = \frac{1}{2}}$$

2. Using Fourier integral formula, prove that $e^{-x}\cos x = \frac{2}{\pi}\int_0^\infty \frac{\lambda^2+2}{\lambda^4+4}\cos\lambda x\,d\lambda$.

Solution:

Given: $f(x) = e^{-x}\cos x \Longrightarrow f(t) = e^{-t}\cos t$

w.k.t. the Fourier cosine integral formula

$$\boldsymbol{f(x) = \frac{2}{\pi}\int_0^\infty \cos\lambda x\,\{\int_0^\infty f(t)\cos\lambda t\,dt\}\,d\lambda} \text{ ------------------ (1)}$$

$$= \frac{2}{\pi}\int_0^\infty \cos\lambda x\,\{\int_0^\infty e^{-t}\cos t\cos\lambda t\,dt\}\,d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \cos\lambda x\,\left\{\int_0^\infty e^{-t}\left(\frac{\cos(1+\lambda)t + \cos(1-\lambda)t}{2}\right)dt\right\}\,d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \cos\lambda x\,\{\int_0^\infty e^{-t}\cos(1+\lambda)t\,dt + \int_0^\infty e^{-t}\cos(1-\lambda)t\,dt\}\,d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \cos\lambda x \left\{\frac{1}{1+(1+\lambda^2)} + \frac{1}{1+(1-\lambda^2)}\right\} d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \cos\lambda x \left\{\frac{1}{\lambda^2+2\lambda+2} + \frac{1}{\lambda^2-2\lambda+2}\right\} d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \cos\lambda x \left\{\frac{\lambda^2-2\lambda+2+\lambda^2+2\lambda+2}{(\lambda^2+2\lambda+2)(\lambda^2-2\lambda+2)}\right\} d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \cos\lambda x \left\{\frac{2\lambda^2+4}{\lambda^4+4}\right\} d\lambda$$

$$\boldsymbol{f(x) = \frac{1}{\pi}\int_0^\infty \frac{2\lambda^2+4}{\lambda^4+4}\cos\lambda x \, d\lambda}$$

3. Using Fourier integral formula, prove that

$$\boldsymbol{e^{-ax} - e^{-bx} = \frac{2(b^2-a^2)}{\pi}\int_0^\infty \frac{\lambda\sin\lambda x}{(\lambda^2+a^2)(\lambda^2+b^2)} d\lambda}$$

Solution:

Given: $f(x) = e^{-ax} - e^{-bx} \implies f(t) = e^{-at} - e^{-bt}$

w.k.t. Fourier sine integral formula

$$\boldsymbol{f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x \left\{\int_0^\infty f(t)\sin\lambda t \, dt\right\} d\lambda} \text{ ---------- } \mathbf{(1)}$$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x \left\{\int_0^\infty (e^{-at} - e^{-bt})\sin\lambda t \, dt\right\} d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x \left\{\int_0^\infty e^{-at}\sin\lambda t \, dt + \int_0^\infty e^{-bt}\sin\lambda t \, dt\right\} d\lambda$$

$$\left\{\because \int \boldsymbol{e^{ax}\sin bx \, dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]}\right\}$$

$$= \frac{2}{\pi}\int_0^\infty \sin\lambda x \left\{\frac{\lambda}{\lambda^2+a^2} - \frac{\lambda}{\lambda^2+b^2}\right\} d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \lambda\sin\lambda x \left\{\frac{\lambda^2+b^2-\lambda^2-a^2}{(\lambda^2+a^2)(\lambda^2+b^2)}\right\} d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \lambda\sin\lambda x \left\{\frac{b^2-a^2}{(\lambda^2+a^2)(\lambda^2+b^2)}\right\} d\lambda$$

$$= \boldsymbol{\frac{2(b^2-a^2)}{\pi}\int_0^\infty \frac{\lambda\sin\lambda x}{(\lambda^2+a^2)(\lambda^2+b^2)} d\lambda}$$

FOURIER TRANSFORM

The infinite Fourier transform of the function $f(x)$ is defined by

$$F[f(x)] = \int_{-\infty}^{\infty} f(x)e^{-isx}dx$$

$$= F(s)$$

The function $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F[f(x)]e^{isx}ds = \bar{F}(s)$ is called the inversion formula for the Fourier transform and it is denoted by $F^{-1}\big[F[f(x)]\big]$.

FOURIER SINE TRANSFORM

The infinite Fourier transform of the function $f(x)$ is defined by

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx\, dx = F_s(s)$$

The inverse Fourier sine transform denoted by $F_s^{-1}[F_s(f(x)]$ is defined by

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s[f(x)] \sin sx\, dx = F_s^{-1}[F_s(f(x))]$$

FOURIER COSINE TRANSFORM

The infinite Fourier transform of the function $f(x)$ is defined by

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx = F_c(s)$$

The inverse Fourier sine transform denoted by $F_s^{-1}[F_s(f(x)]$ is defined by

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c[f(x)] \cos sx\, dx = F_c^{-1}[F_c(f(x))]$$

4. Find the Fourier transform of $f(x)$, defined as $f(x) = \begin{cases} 1 & for\ |x| < a \\ 0 & for\ |x| > a \end{cases}$ and hence find the value of $\int_0^\infty \frac{sinx}{x} dx$.

Solution:

Given: $f(x) = \begin{cases} 1 & for\ |x| < a \\ 0 & for\ |x| > a \end{cases}$

w.k.t the Fourier transform is

$$F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{-isx} dx \text{ ------------------- (1)}$$

$$= \int_{-\infty}^{-a} f(x) e^{-isx} dx + \int_{-a}^{a} f(x) e^{-isx} dx + \int_{a}^{\infty} f(x) e^{-isx} dx$$

$$= \int_{-\infty}^{-a} 0.\, e^{-isx} dx + \int_{-a}^{a} 1.\, e^{-isx} dx + \int_{a}^{\infty} 0.\, e^{-isx} dx$$

$$= \int_{-a}^{a} e^{-isx} dx$$

$$= \left[\frac{e^{-isx}}{-is}\right]_{-a}^{a}$$

$$= \frac{1}{is}\left[e^{-isa} - e^{isa}\right]$$

$$= -\frac{1}{is}[\cos sa - isin\ sa - \cos sa - isin\ sa]$$

$$= -\frac{1}{is}[-2isin\ sa]$$

$$F[f(x)] = \frac{2sinsa}{s} = F(s)$$

Taking inverse Fourier transform is

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F[f(x)] e^{isx} ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2sinsa}{s}\right)e^{isx}ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2sinsa}{s}\right)(\cos sx + isin\, sx)ds$$

$$= \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty}\left(\frac{2sinsa}{s}\right)(\cos sx)ds + \int_{-\infty}^{\infty}\left(\frac{2sinsa}{s}\right)(isin\, sx)ds\right\}$$

$$\left\{\begin{matrix}\because \int_{-\infty}^{\infty}\left(\frac{2sinsa}{s}\right)(isin\, sx)ds = 0 \\ \text{Because it is odd function}\end{matrix}\right\}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{sinsa}{s}\right)(\cos sx)ds$$

$$\boldsymbol{f(x) = \frac{2}{\pi}\int_{0}^{\infty}\left(\frac{sinsa}{s}\right)(\cos sx)ds}$$

Put $x = 0$ and $a = 1$, we have

$$f(0) = \frac{2}{\pi}\int_{0}^{\infty}\left(\frac{sinsa}{s}\right)(\cos 0)ds$$

$$\frac{2}{\pi}\int_{0}^{\infty}\left(\frac{sins}{s}\right)ds = 1$$

$$\int_{0}^{\infty}\left(\frac{sins}{s}\right)ds = \frac{\pi}{2}$$

Changing the dummy variable s into x, we have

$$\boldsymbol{\int_{0}^{\infty}\left(\frac{\sin x}{x}\right)dx = \frac{\pi}{2}}$$

5. Find the inverse Fourier transform of $\bar{f}(s)$ given by $\bar{f}(s) = \begin{cases} a - |s| & for \quad |s| \le a \\ 0 & for \quad |s| > a \end{cases}$.

Hence show that $\int_{0}^{\infty}\frac{\sin^2 x}{x^2}dx = \frac{\pi}{2}$.

Solution:

Given: $\bar{f}(s) = \begin{cases} a - |s| & for \quad |s| \le a \\ 0 & for \quad |s| > a \end{cases}$

w.k.t. the inverse Fourier transform is

$$F^{-1}\left[F[f(x)]\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty}F[f(x)]e^{isx}ds$$

$$= \frac{1}{2\pi}\left\{\int_{-\infty}^{a}F[f(x)]e^{isx}ds + \int_{-a}^{a}F[f(x)]e^{isx}ds + \int_{a}^{\infty}F[f(x)]e^{isx}ds\right\}$$

$$= \frac{1}{2\pi}\left\{\int_{-\infty}^{a}0.e^{isx}ds + \int_{-a}^{a}[a - |s|]e^{isx}ds + \int_{a}^{\infty}0.e^{isx}ds\right\}$$

$$= \frac{1}{2\pi}\left\{\int_{-a}^{a}[a - |s|]e^{isx}ds\right\}$$

$$= \frac{1}{2\pi}\left\{\int_{-a}^{a}[a - s][\cos sx + isin\, sx]ds\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^a [a-s][\cos sx]ds\right\}$$

$$= \frac{1}{\pi}\left[(a-s)\frac{sin\, sx}{x} - \frac{cos\, sx}{x^2} \right]_0^a$$

$$= \frac{1}{\pi}\left[-\frac{\cos sa}{x^2} + \frac{1}{x^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{1-\cos sa}{x^2}\right]$$

$$= \frac{1}{\pi x^2}\left[2\sin^2\frac{ax}{2}\right]$$

$$= \frac{\frac{a^2x^2}{4}}{\pi x^2}\left[\frac{2\sin^2\frac{ax}{2}}{\frac{a^2x^2}{4}}\right]$$

$$\boldsymbol{f(x) = \frac{a^2}{2\pi}\left[\frac{sin\frac{ax}{2}}{\frac{ax}{2}}\right]^2}$$

Taking Fourier transform $\boldsymbol{F[f(x)] = \int_{-\infty}^{\infty} f(x)e^{-isx}dx}$

$$a - s = \int_{-\infty}^{\infty}\frac{a^2}{2\pi}\left[\frac{sin\frac{ax}{2}}{\frac{ax}{2}}\right]^2 e^{-isx}dx$$

$$a - s = \frac{2a^2}{2\pi}\int_0^{\infty}\left[\frac{sin\frac{ax}{2}}{\frac{ax}{2}}\right]^2 [\cos sx - i\sin sx]dx$$

$$a - s = \frac{a^2}{\pi}\int_0^{\infty}\left[\frac{sin\frac{ax}{2}}{\frac{ax}{2}}\right]^2 [\cos sx]dx$$

Put $a = 2$, $s = 0$, we have

$$2 = \frac{4}{\pi}\int_0^{\infty}\left[\frac{sin\, x}{x}\right]^2 dx$$

$$\int_0^{\infty}\left[\frac{sin\, x}{x}\right]^2 dx = \frac{\pi}{2}$$

6. Find the Fourier transform of $e^{-a^2x^2}$. Hence

(i) Prove that $e^{-\frac{x^2}{2}}$ is self reciprocal with respect to Fourier transform.

(ii) Find the Fourier cosine transform of $e^{-\frac{x^2}{2}}$.

Solution:

Given: $f(x) = e^{-a^2x^2}$

w.k.t the Fourier transform is

$$\boldsymbol{F[f(x)] = \int_{-\infty}^{\infty} f(x)e^{-isx}dx} \quad \text{------------------------ (1)}$$

$$= \int_{-\infty}^{\infty} e^{-a^2x^2}e^{-isx}dx$$

$$= \int_{-\infty}^{\infty} e^{-(a^2x^2+isx)}dx$$

$$= \int_{-\infty}^{\infty} e^{-\left(a^2x^2+isx-\frac{i^2s^2}{4a^2}+\frac{i^2s^2}{4a^2}\right)} dx$$

$$= \int_{-\infty}^{\infty} e^{-\left(a^2x^2+isx-\frac{i^2s^2}{4a^2}\right)} . e^{\frac{i^2s^2}{4a^2}} dx$$

$$= \int_{-\infty}^{\infty} e^{-\left(ax+\frac{is}{2a}\right)^2} . e^{-\left(\frac{s^2}{4a^2}\right)} dx$$

$$= e^{-\left(\frac{s^2}{4a^2}\right)} \int_{-\infty}^{\infty} e^{-\left(ax+\frac{is}{2a}\right)^2} . dx$$

Take $t = ax + \frac{is}{2a}$

$dt = adx$

x	$-\infty$	∞
t	$-\infty$	∞

$$= e^{-\left(\frac{s^2}{4a^2}\right)} \int_{-\infty}^{\infty} e^{-t^2} . \frac{dt}{a}$$

$$= e^{-\left(\frac{s^2}{4a^2}\right)} \frac{1}{a} \int_{-\infty}^{\infty} e^{-t^2} dt$$

$$= \frac{\sqrt{\pi}}{a} e^{-\left(\frac{s^2}{4a^2}\right)} \text{ ---------------- (2)}$$

(i) we assumed the definition of the Fourier transform as

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-isx} dx$$

$$(1) \Longrightarrow F\left[e^{-a^2x^2}\right] = \frac{1}{a\sqrt{2}} e^{-\frac{s^2}{4a^2}}$$

Put $= \frac{1}{\sqrt{2}}$, we have

$$= e^{-\frac{s^2}{2}}$$

$$\boldsymbol{F\left[e^{-\frac{x^2}{2}}\right] = e^{-\frac{s^2}{2}} \text{ and } F\left[e^{-\frac{s^2}{2}}\right] = e^{-\frac{x^2}{2}}}$$

(ii) From (2), we have

$$\int_{-\infty}^{\infty} e^{-a^2x^2} (\cos sx - i\sin sx)\, dx = \frac{\sqrt{\pi}}{a} e^{-s^2}/4a^2$$

Equating the real part on both sides, we have

$$\int_{-\infty}^{\infty} e^{-a^2x^2} (\cos sx)\, dx = \frac{\sqrt{\pi}}{a} e^{-s^2}/4a^2$$

$$\boldsymbol{F_c\left[e^{-a^2x^2}\right] = \frac{\sqrt{\pi}}{a} e^{-s^2}/4a^2}$$

TRY YOURSELF

1. Find the Fourier transform of $f(x)$ given by $f(x) = \begin{cases} 1 - x^2 & for\ |x| < 1 \\ 0 & for\ \ |x| > 1 \end{cases}$. Hence evaluate $\int_0^{\infty} \left[\frac{\sin x - x\cos x}{x^3}\right] \cos\frac{x}{2} dx$.

2. Find the Fourier cosine transform of e^{-ax} and use it to find the Fourier transform of $e^{-a|x|}\cos bx$.

3. Find the Fourier cosine transform of $f(x)$ is defined as $f(x) = \begin{cases} 1 & for\ \ 0 < x < a \\ 0 & for\ \ \ \ \ \ \ x \geq a \end{cases}$.

   Hence find the inverse Fourier cosine transform of $\left(\frac{\sin as}{s}\right)$. Verify your answer by directly finding $F_c^{-1}\left[\frac{\sin as}{s}\right]$.

4. Find the Fourier sine transform of $f(x)$ defined as $f(x) = \begin{cases} \sin x & for\ 0 < x < a \\ 0 & for\ \ \ x > a \end{cases}$.

5. Find the Fourier sine transform of $e^{-ax}, a > 0$. Hence find $F_s[xe^{-ax}]$ and $F_s\left[\frac{e^{-ax}}{x}\right]$.

   Deduce the value of $\int_0^\infty \frac{\sin sx}{x}\, dx$.

PROPERTIES OF FOURIER TRANSFORM

1. Linear Property:

$$\boldsymbol{F[\alpha f(x) \pm \beta g(x)] = \alpha F(s) + \beta G(s)} \textbf{ where } \boldsymbol{\alpha} \textbf{ and } \boldsymbol{\beta} \textbf{ are constant.}$$

Proof:

w.k.t. the Fourier transform is given by

$$\boldsymbol{F[f(x)] = \int_{-\infty}^{\infty} f(x)e^{-isx}dx}$$

L.H.S. $\boldsymbol{F[\alpha f(x) \pm \beta g(x)]} = \int_{-\infty}^{\infty}[\alpha f(x) \pm \beta g(x)]e^{-isx}dx$

$$= \int_{-\infty}^{\infty}[\alpha f(x)]e^{-isx}dx \pm \int_{-\infty}^{\infty}[\beta g(x)]e^{-isx}dx$$

$$= \alpha \int_{-\infty}^{\infty} f(x)e^{-isx}dx \pm \beta \int_{-\infty}^{\infty} g(x)e^{-isx}dx$$

$$= \boldsymbol{\alpha\, F(x) \pm \beta G(x)} \ \ \textbf{R.H.S.}$$

2. Change of Scale property:

$$\boldsymbol{F[f(ax)] = \frac{1}{a}F\left(\frac{s}{a}\right), a} > 0$$

Proof:

w.k.t. the Fourier transform is given by

$$\boldsymbol{F[f(x)] = \int_{-\infty}^{\infty} f(x)e^{-isx}dx}$$

L.H.S. $\boldsymbol{F[f(ax)]} = \int_{-\infty}^{\infty} f(ax)e^{-isx}dx$

Take $ax = t \implies x = \frac{t}{a}$

$adx = dt \implies dx = \frac{dt}{a}$

x	$-\infty$	∞
t	$-\infty$	∞

$$= \int_{-\infty}^{\infty} f(t)e^{-is\left(\frac{t}{a}\right)}\frac{dy}{a}$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} f(t)e^{-i\left(\frac{s}{a}\right)t}\, dy$$

$$= \frac{1}{a}F\left(\frac{s}{a}\right), a > 0 \text{ R.H.S.}$$

3. Shifting Property

(i) $F[f(x-a)] = e^{-ias}F(s)$

(ii) $F[e^{iax}f(x)] = F(s-a)$

Proof:

w.k.t. the Fourier transform is given by

$$F[f(x)] = \int_{-\infty}^{\infty} f(x)e^{-isx}dx$$

(i) L.H.S. $F[f(x-a)] = \int_{-\infty}^{\infty} f(x-a)e^{-isx}dx$

Take $x - a = t \implies x = t + a$

$dx = dt$

x	$-\infty$	∞
t	$-\infty$	∞

$$= \int_{-\infty}^{\infty} f(t)e^{-is(t+a)}dt$$

$$= \int_{-\infty}^{\infty} f(t)e^{-ist}.e^{-isa}dt$$

$$= e^{-isa}\int_{-\infty}^{\infty} f(t)e^{-ist}dt$$

$$= e^{-isa}F(s) \text{ R.H.S.}$$

(ii) L.H.S. $F\left[e^{iax}f(x)\right] = \int_{-\infty}^{\infty} e^{iax}f(x)\, e^{-isx}dx$

$$= \int_{-\infty}^{\infty} f(x)\, e^{-i(s-a)x}dx$$

$$= F(s-a) \text{ R.H.S.}$$

4. Modulation Property:

$$F[f(x)\cos ax] = \frac{1}{2}[F(s+a) + F(s-a)]$$

Proof:

w.k.t. the Fourier transform is given by

$$F[f(x)] = \int_{-\infty}^{\infty} f(x)e^{-isx}dx$$

L.H.S. $F[f(x)\cos ax] = \int_{-\infty}^{\infty}(f(x)\cos ax)e^{-isx}dx$

$$= \int_{-\infty}^{\infty} f(x)\left(\frac{e^{iax}+e^{-iax}}{2}\right)e^{-isx}dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} f(x)\left(e^{iax} + e^{-iax}\right)e^{-isx}dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} f(x)e^{iax}e^{-isx}dx + \int_{-\infty}^{\infty} f(x)e^{-iax}e^{-isx}dx\right]$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} f(x)e^{-i(s-a)x}dx + \int_{-\infty}^{\infty} f(x)e^{-i(s+a)x}dx\right]$$

$$= \frac{1}{2}[F(s+a) + F(s-a)] \text{ R.H.S}$$

Results:

(i) $F_c[f(x)cos\ ax] = \frac{1}{2}[F_c(s+a) + F_c(s-a)]$

(ii) $F_c[f(x)sin\ ax] = \frac{1}{2}[F_s(a+s) + F_s(a-s)]$

(iii) $F_S[f(x)cos\ ax] = \frac{1}{2}[F_s(s+a) + F_s(s-a)]$

(iv) $F_c[f(x)sin\ ax] = \frac{1}{2}[F_c(s-a) - F_c(s+a)]$

5. Conjugate Symmetry Property:

$\boldsymbol{F[f^*(-x)] = [F(s)]^*}$**, where** $*$ **denotes the complex conjugate.**

Proof:

w.k.t. the Fourier transform is given by

$$\boldsymbol{F[f(x)] = \int_{-\infty}^{\infty} f(x)e^{-isx}dx = F(s)}$$

R.H.S. $[\boldsymbol{F(s)}]^* = \int_{-\infty}^{\infty} f^*(x)e^{isx}dx$

$= \int_{-\infty}^{\infty} f^*(-t)e^{-ist}dt$ on put $x = -t$

$= \boldsymbol{F[f^*(-x)]}$ **L.H.S.**

6. Derivatives of the Transform:

$$\boldsymbol{F[xf(x)] = \frac{d}{ds}F(s)}$$

Proof:

w.k.t. the Fourier transform is given by

$$\boldsymbol{F(s) = \int_{-\infty}^{\infty} f(x)e^{-isx}dx}$$

Differentiate w.r.t. to s, we have

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_{-\infty}^{\infty} f(x)e^{-isx}dx$$

$$= \int_{-\infty}^{\infty} f(x)\frac{\partial}{\partial s}\left(e^{-isx}\right)dx$$

$$= \int_{-\infty}^{\infty} f(x)\left(e^{-isx}(-ix)\right)dx$$

$$= -i\int_{-\infty}^{\infty} xf(x)\, e^{-isx}dx$$

$$= \boldsymbol{-i.F[xf(x)]}$$

In general, $\boldsymbol{F[x^n f(x)] = (-i)^n \frac{d^n}{ds^n}F(s)}$

Definition:

The convolution of $f(x)$ and $g(x)$ is defined as $\int_{-\infty}^{\infty} f(x-t)g(t)dt$. It is denoted by $f(x) * g(x)$.

CONVOLUTION THEOREM FOR FOURIER TRANSFORMS

Statement: If $F(s)$ and $G(s)$ are the Fourier transform of $f(x)$ and $g(x)$ respectively then the Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transform.

(i.e.,) $F[(f * g)x] = F(s).G(s)$

Proof:

w.k.t. the Fourier transform is given by

$$\boldsymbol{F[f(x)] = \int_{-\infty}^{\infty} f(x)e^{-isx}dx}$$

$$F[(f * g)x] = \int_{-\infty}^{\infty}\big((f * g)x\big)e^{-isx}dx$$

$$= \int_{-\infty}^{\infty}\big[\int_{-\infty}^{\infty} f(t)g(x - t)dt\big]e^{-isx}dx$$

$$= \int_{-\infty}^{\infty} f(t)\big[\int_{-\infty}^{\infty} g(x - t)e^{-isx}dx\big]\,dt$$

on changing the order of integration

$$= \int_{-\infty}^{\infty} f(t)G[g(x - t)]\,dt$$

$$= \int_{-\infty}^{\infty} f(t)e^{ist}G(s)\,dt$$

$$= G(s)\int_{-\infty}^{\infty} f(t)e^{ist}\,dt$$

$$= G(s).F(s)$$

$\therefore$ $\boldsymbol{F[(f * g)x] = F(s).G(s)}$

PARSEVAL'S IDENTITY

Let $F(s)$ be the Fourier transform of $f(x)$. Then $\int_{-\infty}^{\infty}|\boldsymbol{f(x)}|^2\boldsymbol{dx} = \frac{\boldsymbol{1}}{\boldsymbol{2\pi}}\int_{-\infty}^{\infty}|\boldsymbol{F(s)}|^2\boldsymbol{ds}$.

7. **Find the Fourier transform of $f(x)$, if $f(x) = \begin{cases} 1 - |x|\ for\ |x| < 1 \\ 0 \quad\quad for\ |x| > 1 \end{cases}$. Hence prove that $\int_0^{\infty}\frac{\sin^4 x}{x^4}dx = \frac{\pi}{3}$.**

Solution:

Given: $f(x) = \begin{cases} 1 - |x|\ for\ |x| < 1 \\ 0 \quad\quad for\ |x| > 1 \end{cases}$

w.k.t. the Fourier transform is given by

$$\boldsymbol{F[f(x)] = \int_{-\infty}^{\infty} f(x)e^{-isx}dx = F(s)}$$

$$= \int_{-\infty}^{-1} f(x)e^{-isx}dx + \int_{-1}^{1} f(x)e^{-isx}dx + \int_{1}^{\infty} f(x)e^{-isx}dx$$

$$= \int_{-\infty}^{-1} 0.e^{-isx}dx + \int_{-1}^{1}(1 - |x|)e^{-isx}dx + \int_{1}^{\infty} 0.e^{-isx}dx$$

$$= \int_{-1}^{1}(1 - |x|)e^{-isx}dx$$

$$= \int_{-1}^{1}(1-|x|)(\cos sx - i\sin sx)dx$$

$$\left\{\because \int_{-1}^{1}(1-|x|)(i\sin sx)ds = 0 \atop \text{Because it is odd function}\right\}$$

$$= \int_{-1}^{1}(1-x)(\cos sx)dx$$

$$= 2\int_{0}^{1}(1-x)(\cos sx)dx$$

$$= 2\left[(1-x)\frac{\sin sx}{s} - \frac{\cos sx}{s^2}\right]_0^1$$

$$= 2\left[-\frac{\cos s}{s^2} + \frac{1}{s^2}\right]$$

$$= \frac{2}{s^2}[1-\cos s]$$

$$= \frac{2}{s^2}\left[2\sin^2\frac{s}{2}\right]$$

$$\boldsymbol{F(s) = \frac{4}{s^2}\left[\sin^2\frac{s}{2}\right]}$$

By Parseval's Identity, $\int_{-\infty}^{\infty}|\boldsymbol{f(x)}|^2\boldsymbol{dx} = \frac{\boldsymbol{1}}{\boldsymbol{2\pi}}\int_{-\infty}^{\infty}|\boldsymbol{F(s)}|^2\boldsymbol{ds}$

$$\int_{-1}^{1}|1-|x||^2dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{4}{s^2}\left[\sin^2\frac{s}{2}\right]\right|^2 ds$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{4}{s^2}\left[\sin^2\frac{s}{2}\right]\right]^2 ds = \int_{-1}^{1}[1-x]^2dx$$

$$\frac{16}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{s^2}\left[\sin^2\frac{s}{2}\right]\right]^2 ds = 2\left[\frac{(1-x)}{-3}\right]_0^1$$

$$\frac{8}{\pi}\int_{-\infty}^{\infty}\frac{1}{s^4}\left[\sin^4\frac{s}{2}\right]ds = \frac{2}{3}$$

Take $\frac{s}{2} = t \quad \Rightarrow s = 2t$

$$\Rightarrow ds = 2dt$$

s	$-\infty$	∞
t	$-\infty$	∞

$$\frac{8}{\pi}\int_{-\infty}^{\infty}\frac{1}{(2t)^4}[\sin^4 t]\,(2dt) = \frac{2}{3}$$

$$\frac{8}{\pi}\int_{-\infty}^{\infty}\left[\frac{\sin^4 t}{16t^4}\right](2dt) = \frac{2}{3}$$

$$\frac{2}{\pi}\int_{0}^{\infty}\left[\frac{\sin^4 t}{t^4}\right]dt = \frac{2}{3}$$

$$\boldsymbol{\int_{0}^{\infty}\left[\frac{\sin^4 t}{t^4}\right]dt = \frac{\pi}{3}}$$

8. Using the Parseval's Identity for Fourier cosine transform of e^{-ax} . Show that $\int_0^{\infty}\frac{dx}{(x^2+a^2)^2}$.

Solution:

Given: $f(x) = e^{-ax}$

w.k.t. Fourier cosine transform is

$$\boldsymbol{F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\, dx = F_c(s)}$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{a}{s^2+a^2}\right]$$

By Parseval's Identity, $\int_{-\infty}^{\infty}|f(x)|^2 dx = \int_{-\infty}^{\infty}|F(s)|^2 ds$

$$\int_{-\infty}^{\infty}[e^{-ax}]^2 dx = \int_{-\infty}^{\infty}\left[\sqrt{\frac{2}{\pi}}\left[\frac{a}{s^2+a^2}\right]\right]^2 ds$$

$$2\left(\frac{2}{\pi}\right)\int_0^\infty\left(\frac{a^2}{(s^2+a^2)^2}\right)ds = 2\int_0^\infty e^{-2ax}dx$$

$$\frac{2}{\pi}\int_0^\infty\left(\frac{a^2}{(s^2+a^2)^2}\right)ds = \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty$$

$$\frac{2a^2}{\pi}\int_0^\infty\left(\frac{1}{(s^2+a^2)^2}\right)ds = \frac{1}{2a}$$

$$\int_0^\infty\left(\frac{1}{(s^2+a^2)^2}\right)ds = \frac{\pi}{4a^3}$$

By changing the dummy variable s into x, we have

$$\int_0^\infty\left(\frac{1}{(x^2+a^2)^2}\right)dx = \frac{\pi}{4a^3}$$

TRY YOURSELF

1. Using the Parseval's Identity for Fourier sine transform of e^{-ax} . Show that $\int_0^\infty \frac{x^2 dx}{(x^2+a^2)^2}$.

UNIT – IV

LAPLACE TRANSFORM

DEFINITION:

Let $f(t)$ be a function defined in the interval $0 \leq t \leq \infty$. Then the Laplace transform of $f(t)$ is given by $\int_0^\infty e^{-st} f(t)dt$. It is denoted by $L[f(t)]$ which is a function of s say $f(s)$.

$$\therefore f(s) = L[f(t)] = \int_0^\infty e^{-st} f(t)dt$$

PROPERTIES:

1. $\boldsymbol{L[f(t) + g(t)] = L[f(t)] + L[g(t)]}$

Proof: $L[f(t) + g(t)] = \int_0^\infty e^{-st}(f(t) + g(t))dt$

$$= \int_0^\infty e^{-st} f(t)dt + \int_0^\infty e^{-st} g(t)dt$$

$$= L[f(t)] + L[g(t)].$$

2. $\boldsymbol{L[cf(t)] = c\,L[f(t)]}$

Proof: $L[cf(t)] = \int_0^\infty e^{-st} cf(t)dt$

$$= c \int_0^\infty e^{-st} f(t)dt$$

$$= c\,L[f(t)].$$

3. $\boldsymbol{L[f'(t)] = s\,L[f(t)] - f(0)}$

Proof: $L[f'(t)] = \int_0^\infty e^{-st} f'^{(t)} dt$

$$= \int_0^\infty e^{-st} d(f(t))$$

$$= [e^{-st} f(t)]_0^\infty - \int_0^\infty -se^{-st} f(t)dt$$

$$= [e^{-\infty} f(\infty) - e^0 f(0)] + s \int_0^\infty e^{-st} f(t)dt$$

$$= -f(0) + s\,L[f(t)]$$

4. $\boldsymbol{L[f''(t)] = s^2L[f(t)] - sf(0) - f'(0)}$

Proof: Let $g(t) = f'(t)$

$$L[f''(t)] = L[g'(t)]$$

$$= sL[g(t)] - g(0) \quad \text{(by Property. 3)}$$

$$= sL[f'(t)] - f'(0)$$

$$= s(sL[f(t)] - f(0)) - f'(0) \quad \text{(by Property. 3)}$$

$$= s^2L[f(t)] - sf(0) - f'(0).$$

Initial Value Theorem

If $L[f(t)] = F[s]$ then $lim_{t\to 0} f(t) = lim_{s\to\infty} sF[s]$.

Proof:

w.k.t. $L[f'(t)] = s\,L[f(t)] - f(0) = s\,F[s] - f(0)$

Taking limit as $s \to \infty$ on both sides, we get

$$\lim_{s\to\infty} L[f'(t)] = \lim_{s\to\infty}[s\,F[s] - f(0)]$$

$$\lim_{s\to\infty} \int_0^\infty e^{-st} f'(t)dt = \lim_{s\to\infty} sF[s] - f(0)$$

$$0 = \lim_{s\to\infty} sF[s]$$

Thus, $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF[s]$.

FINAL VALUE THEOREM

If $L[f(t)] = F[s]$ then $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF[s]$.

Proof. W.K.T. $L[f'(t)] = s\,L[f(t)] - f(0) = s\,F[s] - f(0)$

Taking limit as $s \to 0$ on both sides, we get

$$\lim_{s\to 0} L[f'(t)] = \lim_{s\to 0}[s\,F[s] - f(0)]$$

$\lim_{s\to 0} \int_0^\infty e^{-st} f'(t)dt = \lim_{s\to 0} sF[s] - f(0)$

$\int_0^\infty f'(t)dt = \lim_{s\to 0} sF[s] - f(0)$

$[f(t)]_0^\infty = \lim_{s\to 0} sF[s] - f(0)$

$f(\infty) - f(0) = \lim_{s\to 0} sF[s] - f(0)$

$f(\infty) = \lim_{s\to 0} sF[s]$

Thus, $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF[s]$.

RESULTS:

1. $\boldsymbol{L[e^{-at}] = \frac{1}{s+a}}$

Proof:

w.k.t. the Laplace transform is

$$\boldsymbol{L[f(t)] = \int_0^\infty e^{-st} f(t)dt}$$

$$L[e^{-at}] = \int_0^\infty e^{-st} e^{-at} dt$$

$$= \int_0^\infty e^{-(s+a)t} dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

$$= \frac{e^{-\infty}}{-(s+a)} + \frac{e^0}{(s+a)}$$

$$= \boldsymbol{\frac{1}{s+a}}$$

Similarly, $\boldsymbol{L[e^{at}] = \frac{1}{s-a}}$

2. $\boldsymbol{L[\cos at] = \frac{s}{s^2+a^2}}$

Proof:

w.k.t. the Laplace transform is

$$\boldsymbol{L[f(t)] = \int_0^\infty e^{-st} f(t)dt}$$

$$L\left[\frac{e^{iat}+e^{-iat}}{2}\right] = \frac{1}{2} L\left[e^{iat} + e^{-iat}\right] \qquad \left\{\because \cos x = \frac{e^{ix}+e^{-ix}}{2i}\right\}$$

$$= \frac{1}{2}\left\{L\left[e^{iat}\right] + L\left[e^{-iat}\right]\right\}$$

$$= \frac{1}{2}\left[\frac{1}{s-ia} + \frac{1}{s+ia}\right] \qquad \{\because by\ Result\ 1\}$$

$$= \frac{1}{2}\left[\frac{s+ia+s-ia}{(s-ia)(s+ia)}\right]$$

$$= \frac{1}{2}\left[\frac{2s}{(s-ia)(s+ia)}\right]$$

$$= \frac{s}{s^2+a^2}$$

3. $L[\sin at] = \frac{a}{s^2+a^2}$.

Proof:

w.k.t. the Laplace transform is

$$L[f(t)] = \int_0^\infty e^{-st} f(t)dt$$

$$L\left[\frac{e^{iat}-e^{-iat}}{2i}\right] = \frac{1}{2i}L\left[e^{iat} - e^{-iat}\right] \qquad \left\{\because \sin x = \frac{e^{ix}-e^{-ix}}{2i}\right\}$$

$$= \frac{1}{2i}\left\{L\left[e^{iat}\right] - L\left[e^{-iat}\right]\right\}$$

$$= \frac{1}{2i}\left[\frac{1}{s-ia} - \frac{1}{s+ia}\right] \qquad \{\because by\ Result\ 1\}$$

$$= \frac{1}{2i}\left[\frac{s+ia-s+ia}{(s-ia)(s+ia)}\right]$$

$$= \frac{1}{2i}\left[\frac{2ia}{(s-ia)(s+ia)}\right]$$

$$= \frac{a}{s^2+a^2}$$

4. $L[\cosh at] = \frac{s}{s^2-a^2}$.

Proof:

w.k.t. the Laplace transform is

$$L[f(t)] = \int_0^\infty e^{-st} f(t)dt$$

$$L\left[\frac{e^{at}+e^{-at}}{2}\right] = \frac{1}{2}L[e^{at} + e^{-at}] \qquad \left\{\because \cosh x = \frac{e^{x}+e^{-x}}{2}\right\}$$

$$= \frac{1}{2}\{L[e^{at}] + L[e^{-at}]\}$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] \qquad \{\because by\ Result\ 1\}$$

$$= \frac{1}{2}\left[\frac{s+a+s-a}{(s-a)(s+a)}\right]$$

$$= \frac{1}{2}\left[\frac{2s}{(s-a)(s+a)}\right]$$

$$= \frac{s}{s^2-a^2}$$

5. $L[\sinh at] = \frac{a}{s^2-a^2}$.

Proof:

w.k.t. the Laplace transform is

$$L[f(t)] = \int_0^\infty e^{-st} f(t)dt$$

$L\left[\frac{e^{at}-e^{-at}}{2}\right] = \frac{1}{2}L[e^{at} - e^{-at}]$ $\left\{\because \sinh x = \frac{e^{x}-e^{-x}}{2}\right\}$

$= \frac{1}{2}\{L[e^{at}] - L[e^{-at}]\}$

$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$ $\{\because \textit{by Result } 1\}$

$= \frac{1}{2}\left[\frac{s+a-s+a}{(s-a)(s+a)}\right]$

$= \frac{1}{2}\left[\frac{2a}{(s-a)(s+a)}\right]$

$= \frac{\boldsymbol{a}}{\boldsymbol{s^2-a^2}}.$

6. $\boldsymbol{L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}}.$

Proof:

w.k.t. the Laplace transform is

$$\boldsymbol{L[f(t)] = \int_0^\infty e^{-st} f(t)dt}$$

$L[t^n] = \int_0^\infty e^{-st} t^n dt$

$= \left[t^n \frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty n t^{n-1} \frac{e^{-st}}{-s} dt$

$= 0 + \frac{n}{s}\int_0^\infty t^{n-1} e^{-st} dt$

$= \frac{n}{s}\left[t^{n-1}\frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty (n-1) t^{n-2} \frac{e^{-st}}{-s} dt$

$= \frac{n}{s}\frac{n-1}{s}\int_0^\infty t^{n-2} e^{-st} dt$

$= \frac{n}{s}\frac{n-1}{s}\frac{n-2}{s} \ldots \frac{1}{s}\int_0^\infty t^{0} e^{-st} dt$

$= \frac{n!}{s^n}\left[\frac{e^{-st}}{-s}\right]_0^\infty$

$= \frac{n!}{s^n}\left[\frac{0-1}{-s}\right]$

$= \frac{\boldsymbol{n!}}{\boldsymbol{s^{n+1}}}$ **(or)** $\frac{\boldsymbol{\Gamma(n+1)}}{\boldsymbol{s^{n+1}}}$ $\{\because n! = \boldsymbol{\Gamma(n+1)}\}$

7. $\boldsymbol{L(1) = \frac{1}{s}, L(t) = \frac{1}{s^2}, L(t^2) = \frac{2}{s^3}}.$

Proof:

w.k.t. the result 6 is $\boldsymbol{L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}}$

Put $n = 0$, we have $L[t^0] = \frac{0!}{s^{0+1}} = \frac{1}{s}$

Put $n = 1$, we have $L[t] = \frac{1!}{s^{1+1}} = \frac{1}{s^2}$

Put $n = 2$, we have $L[t^2] = \frac{2!}{s^{2+1}} = \frac{2}{s^3}$

8. $L\left[t^{\frac{1}{2}}\right] = \frac{\sqrt{\pi}}{2s^{3/2}}$ and $L\left[t^{-\frac{1}{2}}\right] = \frac{\sqrt{\pi}}{s^{1/2}}$.

Proof:

w.k.t. the Laplace transform is

$$\boldsymbol{L[f(t)] = \int_0^\infty e^{-st} f(t) dt}$$

$$L\left[t^{\frac{1}{2}}\right] = \int_0^\infty e^{-st}\left(t^{\frac{1}{2}}\right) dt$$

Take $st = x \Longrightarrow t = \frac{s}{x}$

$sdt = dx$

t	0	∞
x	0	∞

$$L\left[t^{\frac{1}{2}}\right] = \int_0^\infty e^{-x}\left(\left(\frac{s}{x}\right)^{\frac{1}{2}}\right)\frac{dx}{s}$$

$$= \frac{1}{s.s^{-\frac{1}{2}}}\int_0^\infty e^{-x}x^{-\frac{1}{2}}dx \qquad \left\{\because \Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx = (n-1)!\right\}$$

$$= \frac{1}{\sqrt{s}}\Gamma\left(\frac{1}{2}\right)$$

$$= \sqrt{\frac{\pi}{s}}.$$

PROBLEM

9. Find $\boldsymbol{L(t^2 + 2t + 3)}$.

Solution:

$$L[t^2 + 2t + 3] = L[t^2] + 2l[t] + 3$$

$$= \frac{2}{s^3} + \frac{2}{s^2} + \frac{3}{s}$$

10. Find $\boldsymbol{L[e^{-at} \sin bt]}$.

Solution:

$$L[e^{-at} \sin bt] = \int_0^\infty e^{-st}e^{-at} \sin bt\, dt$$

$$= \int_0^\infty e^{-(a+s)t} \sin bt\, dt$$

$$= \left[\frac{e^{-(s+a)t}}{-((s+a)^2+b^2)}(-(s+a)\sin bt - b\cos bt)\right]_0^\infty$$

$$= \left[0 - \frac{e^0}{(s+a)^2+b^2}(-(s+a)\sin 0 - b\cos 0)\right]$$

$$= 0 - \frac{1}{(s+a)^2+b^2}(0 - b)$$

$$= \frac{\boldsymbol{b}}{(\boldsymbol{s}+\boldsymbol{a})^2+\boldsymbol{b}^2}.$$

11. Find $L[e^{-at}\cos bt]$.

Solution:

$$L[e^{-at}\cos bt] = \int_0^\infty e^{-st}e^{-at}\cos bt\, dt$$

$$= \int_0^\infty e^{-(a+s)t}\cos bt\, dt$$

$$= \left[\frac{e^{-(s+a)t}}{-((s+a)^2+b^2)}(-(s+a)\cos bt + b\sin bt)\right]_0^\infty$$

$$= \left[0 - \frac{e^0}{(s+a)^2+b^2}(-(s+a)\cos 0 - b\sin 0)\right]$$

$$= 0 - \frac{1}{(s+a)^2+b^2}(-(s+a) - b)$$

$$= \frac{\boldsymbol{s+a}}{\boldsymbol{(s+a)^2+b^2}}$$

12. Find $L[\sin^2 2t]$.

Solution:

w.k.t. $\sin^2 2t = \frac{1-\cos 4t}{2}$

$$L[\sin^2 2t] = L\left[\frac{1-\cos 4t}{2}\right]$$

$$= \frac{1}{2}(L[1] - L[\cos 4t])$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4^2}\right]$$

$$= \frac{1}{2}\left[\frac{4^2}{s(s^2+16)}\right]$$

$$= \frac{\boldsymbol{8}}{\boldsymbol{s(s^2+16)}}$$

13. Find $L[\sin^3 t]$.

Solution:

w.k.t. $\sin^3 t = \frac{(3\sin t - \sin 3t)}{4}$

$$L[\sin^3 t] = \frac{3}{4}L[\sin t] - \frac{1}{4}L[\sin 3t]$$

$$= \frac{3}{4}\left(\frac{1}{s^2+1^2}\right) - \frac{3}{s^2+3^2}$$

$$= \frac{3}{4}\left[\frac{s^2+3^2-(s^2+1^2)}{(s^2+1^2)(s^2+3^2)}\right]$$

$$= \frac{3}{4}\left[\frac{8}{(s^2+1^2)(s^2+3^2)}\right]$$

$$= \frac{\mathbf{6}}{(s^2+1^2)(s^2+3^2)}.$$

14. Find $L[f(t)]$ if $f(t) = t^2 + \cos 2t\cos t + \sin^2 2t.$

Solution:

w.k.t. $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$$L[\cos 2t \cos t] = \frac{1}{2}\left(L[\cos 3t] + L[\cos t]\right)$$

$$= \frac{1}{2}\left(\frac{s}{s^2+3^2} + \frac{s}{s^2+1^2}\right)$$

$$= \frac{1}{2}\frac{2s^3+10s}{(s^2+1^2)(s^2+3^2)}$$

$$= \frac{s^3+5s}{(s^2+1^2)(s^2+3^2)}$$

$$L[t^2] = \frac{2}{s^3}, L[\sin^2 2t] = \frac{8}{s(s^2+16)}$$

$$L[f(t)] = L[t2] + L[\cos 2t\cos t] + L[sin22t]$$

$$= \frac{2}{s^3} + \frac{s^3+5s}{(s^2+1^2)(s^2+3^2)} + \frac{8}{s(s^2+16)}.$$

15. Find $L[f(t)]$ where $f(t) = \begin{cases} 0 & 0 < t < 2 \\ 3 & t > 2 \end{cases}.$

Solution:

w.k.t. the Laplace transform is

$$L[f(t)] = \int_0^\infty e^{-st} f(t)dt$$

$$= \int_0^2 e^{-st} f(t)dt + \int_2^\infty e^{-st} f(t)dt$$

$$= \int_0^2 e^{-st}.0\, dt + \int_2^\infty e^{-st} 3\, dt$$

$$= 3\int_2^\infty e^{-st} dt$$

$$= 3\left[\frac{e^{-st}}{-s}\right]_2^{\infty}$$

$$= \frac{3}{s}e^{-2s}.$$

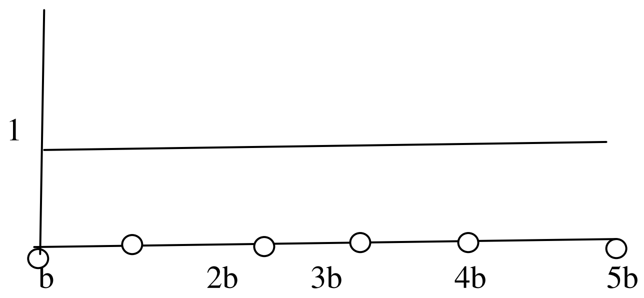
TRY YOURSELF

1. Find $L[f(t)]$ where $f(t) = \begin{cases}(t-1)^2 & t<1 \\ 0 & t>1\end{cases}$.

2. Find $L[f(t)]$ where $f(t) = \begin{cases}\cos t & 0<t<\pi \\ \sin t & t>\pi\end{cases}$.

LAPLACE TRANSFORM OF PERIODIC FUNCTION

1. Find the transform of the rectangular wave as shown below.



Solution:

Given: $f(t) = \begin{cases}1 & 0<t<b \\ -1 & b<t<2b\end{cases}$

The function is periodic in the interval $(0, 2b)$

$$f(t) = \frac{1}{1-e^{-2bs}}\int_0^{2b} e^{-st}f(t)dt$$

$$= \frac{1}{1-e^{-2bs}}\left[\int_0^{b} e^{-st}f(t)dt + \int_b^{2b} e^{-st}f(t)dt\right]$$

$$= \frac{1}{1-e^{-2bs}}\left[\int_0^{b} e^{-st}(1)dt + \int_b^{2b} e^{-st}(-1)dt\right]$$

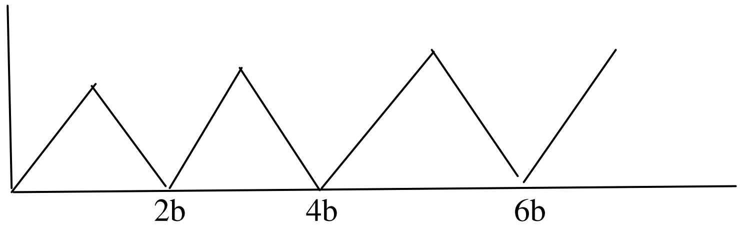
$$= \frac{1}{1-e^{-2bs}}\left(\left[\frac{e^{-st}}{-s}\right]_0^{b} - \left[\frac{e^{-st}}{-s}\right]_b^{2b}\right)$$

$$= \frac{1}{s}\frac{1-2e^{-bs}+e^{-2bs}}{1-e^{-2bs}}$$

$= \frac{1}{s}\frac{1-e^{-bs}}{1-e^{-2bs}}$

$= \frac{\mathbf{1}}{\boldsymbol{s}} \mathbf{tan}\, \boldsymbol{h} \left(\frac{\boldsymbol{bs}}{\mathbf{2}}\right).$

2. What is the transform of the function shown below.



Solution:

Given the function can be represented as

$$f(t) = \begin{cases} t & 0 < t < b \\ 2b - t & b < t < 2b \end{cases}$$

$$f(t) = \frac{1}{1-e^{-2bs}} \int_0^{2b} e^{-st} f(t) dt$$

$$= \frac{1}{1-e^{-2bs}} \left[\int_0^b e^{-st} f(t) dt + \int_b^{2b} e^{-st} f(t) dt \right]$$

$$= \frac{1}{1-e^{-2bs}} \left[\int_0^b e^{-st} (t) dt + \int_b^{2b} e^{-st} (2b - t) dt \right]$$

$$= \frac{\mathbf{1}}{\boldsymbol{s}^2} \mathbf{tan}\, \boldsymbol{h} \left(\frac{\boldsymbol{bs}}{\mathbf{2}}\right).$$

3. Find L[$\boldsymbol{te^{-at}}$].

Solution:

$$\text{L}[te^{-at}] = -\frac{d}{ds}\left(L(e^{-at})\right)$$

$$= -\frac{d}{ds}\left(\frac{1}{s+a}\right)$$

$$= \frac{\mathbf{1}}{(\boldsymbol{s}+\boldsymbol{a})^2}.$$

4. Find $L(t^2 e^{-3t})$

Solution:

$$L(t^2 e^{-3t}) = (-1)^2 \frac{d^2}{ds^2} L(e^{-3t})$$

$$= \frac{d^2}{ds^2}\left(\frac{1}{s+3}\right)$$

$$= \frac{d}{ds}\left(-\frac{1}{(s+3)^2}\right)$$

$$= \frac{\mathbf{2}}{(\boldsymbol{s}+\mathbf{3})^{\mathbf{3}}} .$$

5. Find $L[x^2 cos\ hax]$.

Solution: $L[x^2 cos\ hax] = (-1)^2 \frac{d^2}{ds^2} L(cos\ hax)$

$$= \frac{d}{ds}\left(\frac{d}{ds}\left(\frac{s}{s^2-a^2}\right)\right)$$

$$= \frac{d}{ds}\left(\frac{(s^2-a^2).1-s(2s)}{(s^2-a^2)^2}\right)$$

$$= -\frac{d}{ds}\frac{(s^2+a^2)}{(s^2-a^2)^2}$$

$$= -\frac{(s^2-a^2)^2(2s)-(s^2+a^2)2\left((s^2-a^2)\right)(2s)}{(s^2-a^2)^4}$$

$$= -(s^2-a^2)(2s)\frac{(s^2-a^2)-(2s^2+2a^2)}{(s^2-a^2)^4}$$

$$=-(\mathbf{2s})\frac{-(\boldsymbol{s}^2+\mathbf{3}\boldsymbol{a}^2)}{(\boldsymbol{s}^2-\boldsymbol{a}^2)^{\mathbf{3}}}.$$

6. Find $L[tsin\ at]$

Solution:

$$L[tsin\ at] = -\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right) = \left(\frac{2as}{(s^2+a^2)^2}\right)$$

7. **Find L[$te^{-t}sint$].**

Solution:

$$L[te^{-t}sint] = -\frac{d}{ds}\left(L(e^{-t}sint)\right) = -\frac{d}{ds}F(s+1)$$

Where $F(s) = L(sin\ t) = \frac{1}{s^2+1}$.

$$L[te^{-t}sint] = -\frac{d}{ds}\frac{1}{(s+1)^2+1} = \frac{2(s+1)}{(s^2+2s+2)^2}.$$

8. **If $L[f(t)] = F(s)$ and if $\frac{f(t)}{t}$ has limit as $t \to 0$ then $L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)ds$.**

Solution:

$$F(s) = L[f(t)] = \int_0^\infty e^{-st}f(t)dt$$

$$\int_s^\infty F(s)ds = \int_s^\infty \int_0^\infty e^{-st}f(t)dtds = \int_0^\infty \int_s^\infty e^{-st}f(t)dsdt$$

$$= \int_0^\infty f(t)\left[\frac{e^{-st}}{t}\right]_s^\infty dt = \int_0^\infty \frac{f(t)}{t}e^{-st}dt$$

$$= L\left[\frac{f(t)}{t}\right]$$

9. **Find $L\left[\frac{1-e^t}{t}\right]$.**

Solution:

Here $f(t) = 1 - t$

Now, $\lim_{t\to 0}\frac{1-e^t}{t} = \lim_{t\to 0}\frac{1}{t}\left[1-\left(1+\frac{t}{1!}+\frac{t^2}{2!}+\cdots\right)\right]$

$$= \lim_{t\to 0}\left[-\frac{1}{1!}-\frac{t}{2!}-\cdots\right]$$

$$= -1$$

W.K.T. L[f(t)] = F(s) and if $\frac{f(t)}{t}$ has limit as $t \to 0$ then L[$\frac{f(t)}{t}$] = $\int_s^\infty F(s)ds$.

$$L\left[\frac{1-e^t}{t}\right] = \int_s^\infty L(1-e^t)ds$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1}\right)ds$$

$$= [\log s - log\ s - 1]_s^\infty$$

$$= \left[\log \frac{s}{s-1}\right]_s^\infty$$

$$= 0 \ - log\frac{s}{s-1}$$

$$= \mathbf{log}\frac{\boldsymbol{s-1}}{\boldsymbol{s}}$$

10. Find $L\left[\frac{sin\,at}{t}\right]$.

Solution:

$$L\left[\frac{sin\,at}{t}\right] = \int_s^\infty L(sin\,at)ds = \int_s^\infty \frac{a}{s^2+a^2}ds$$

$$= \left[tan^{-1}\left(\frac{s}{a}\right)\right]_s^\infty$$

$$= tan^{-1}\infty \ - tan^{-1}\left(\frac{s}{a}\right)$$

$$= \frac{\pi}{2} - tan^{-1}\left(\frac{s}{a}\right)$$

$$= \boldsymbol{cot^{-1}}\left(\frac{\boldsymbol{s}}{\boldsymbol{a}}\right).$$

11. Evaluate $\int_0^\infty e^{-2t} \sin 3tdt$.

Solution:

w.k.t. $\int_0^\infty e^{-st} \sin atdt = L(\sin at) = \frac{a}{s^2+a^2}$

Put $s = 2$ and $a = 3$ we get

$\int_0^\infty e^{-2t} \sin 3t dt = \frac{3}{2^2+3^2} = \frac{\mathbf{3}}{\mathbf{13}}$.

12. Evaluate $\int_0^\infty te^{-3t} cost\, dt$.

Solution:

Take $f(t) = cost$

$L[f(t)] = L[cos\, t] = \frac{s}{s^2+1}$

$L[tcost] = (-1)^1 \frac{d}{ds}\left(\frac{s}{s^2+1}\right)$

$= -\frac{(s^2+1).1-s(2s)}{(s^2+1)^2}$

$= \frac{(s^2-1)}{(s^2+1)^2}$

$L[e^{-3t}(tcost)] = \frac{((s+3)^2-1)}{((s+3)^2+1)^2}$

$\int_0^\infty te^{-3t} cost\, dt = L[e^{-3t}(tcost)] = \frac{8}{100}$ [put $s = 0$]

13. Evaluate $\int_0^\infty \frac{e^{-t}-e^{-2t}}{t} dt$.

Solution:

$\int_0^\infty e^{-st} \frac{e^{-t}-e^{-2t}}{t} dt = L\left[\frac{e^{-t}-e^{-2t}}{t}\right]$

$= \int_s^\infty L(e^{-t}) - L(e^{-2t}) ds$

$= \int_s^\infty (\frac{1}{s+1} - \frac{1}{s+2}) ds$

$= \log \frac{s+2}{s+1}$

Put $s = 0$ we get

$\int_0^\infty e^{-st} \frac{e^{-t}-e^{-2t}}{t} dt = \mathbf{\log 2}$.

UNIT V

INVERSE LAPLACE TRANSFORMS

DEFINITION:

Let $f(x)$ is continuous and $L[f(x)] = F(s)$ we have $L^{-1}[F(s)] = f(x)$ and $f(x)$ is called the inverse Laplace transform. Since L is linear and L^{-1} is also linear.

RESULTS:

1. $L^{-1}\left[\frac{1}{s}\right] = 1$ 2. $L^{-1}\left[\frac{1}{s^2}\right] = x$

3. $L^{-1}\left[\frac{1}{s-a}\right] = e^{ax}$ 4. $L^{-1}\left[\frac{1}{s^2+a^2}\right] = sin\ ax$

4. $L^{-1}\left[\frac{s}{s^2+a^2}\right] = cos\ ax$ 5. $L^{-1}\left[\frac{a}{s^2-a^2}\right] = sinh\ ax$

5. $L^{-1}\left[\frac{s}{s^2-a^2}\right] = cosh\ ax$ 6. $L^{-1}\left[\frac{1}{(s-a)^2}\right] = xe^{ax}$

1. Find the inverse Laplace transform of $\frac{s}{s^2a^2+b^2}$.

Solution:

Given $F(s) = \frac{s}{s^2a^2+b^2}$

$$= \frac{1}{a}\left[\frac{sa}{s^2a^2+b^2}\right] = \frac{1}{a}f(sa)$$

Where, $f(sa) = \frac{sa}{s^2a^2+b^2}$. So, $f(s) = \frac{s}{s^2+b^2}$

Now, $L^{-1}\left[\frac{s}{s^2a^2+b^2}\right]=\frac{1}{a}L^{-1}\left[\frac{sa}{s^2a^2+b^2}\right] = \frac{1}{a}L^{-1}[f(sa)] = \frac{1}{a}\times\frac{1}{a}\times f\left(\frac{t}{a}\right)$

Where $f(t) = L^{-1}[f(s)] = L^{-1}\left[\frac{s}{s^2+b^2}\right] = cos\ bt$

$$f\left(\frac{t}{a}\right) = cos\left(\frac{bt}{a}\right).$$

Hence $L^{-1}\left[\frac{s}{s^2a^2+b^2}\right] = \mathbf{\frac{1}{a^2}\cos\left(\frac{bt}{a}\right)}$.

Property 1: If $L[f(t)] = F(s)$ then $L[\,t\,f(t)] = -F'(s)$.

Property 2: $L^{-1}[F'(s)] = -tL^{-1}[F(s)]$

2. Find inverse Laplace transform of $\frac{s}{(s^2+a^2)^2}$.

Solution:

Let $F'(s) = \frac{s}{(s^2+a^2)^2}$

$$F(s) = \int \left[\frac{s}{(s^2+a^2)^2}\right] ds = -\frac{1}{2}\frac{1}{s^2+a^2}$$

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = -tL^{-1}\left[-\frac{1}{2}\frac{1}{s^2+a^2}\right]$$

$$= \frac{t}{2}L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{t}{2a}L^{-1}\left[\frac{a}{s^2+a^2}\right]$$

$$= \frac{t}{2a}\sin at.$$

3. Find inverse Laplace transform of $\frac{s}{(s^2-1)^2}$.

Solution:

Let $F'(s) = \frac{s}{(s^2-1)^2}$

$$F(s) = \int \left[\frac{s}{(s^2-1)^2}\right] ds = -\frac{1}{2(s^2-1)}$$

$$L^{-1}\left[\frac{s}{(s^2-1)^2}\right] = -tL^{-1}\left[-\frac{1}{2(s^2-1)}\right]$$

$$= \frac{t}{2}L^{-1}\left[\frac{1}{s^2-1}\right]$$

$$= \frac{t}{2}\sin ht.$$

4. Find $L^{-1}\left[\frac{2(s+2)}{(s^2+4s+5)^2}\right]$.

Solution:

Let $F'(s) = \frac{2(s+2)}{(s^2+4s+5)^2}$

$F(s) = \int \left[\frac{2(s+2)}{(s^2+4s+5)^2}\right] ds = -\int d(\frac{1}{s^2+4s+5})$

$F(s) = \frac{-1}{s^2+4s+5}$

$L^{-1}\left[\frac{2(s+2)}{(s^2+4s+5)^2}\right] = -tL^{-1}\left[\frac{1}{s^2+4s+5}\right]$

$= tL^{-1}\left[\frac{1}{(s+2)^2+1^2}\right]$

$= te^{-2t}\sin t$

5. Find $L^{-1}\left[log\left(\frac{s+a}{s+b}\right)\right]$.

Solution:

Let $f(t) = L^{-1}\left[log\left(\frac{s+1}{s-1}\right)\right]$

$L[f(t)] = log\left(\frac{s+1}{s-1}\right) = F(s)$

$L[tf(t)] = -F'(s) = -\frac{d}{ds} log\left(\frac{s+1}{s-1}\right)$

$= -\frac{d}{ds}[\log(s+1) - \log(s-1)]$

$= -\left[\frac{1}{s+1} - \frac{1}{s-1}\right]$

$tf(t) = -L^{-1}\left[\frac{1}{s+1}\right] + L^{-1}\left[\frac{1}{s-1}\right]$

$= -e^{-t}L^{-1}\left[\frac{1}{s}\right] + e^{t}L^{-1}\left[\frac{1}{s}\right]$

$= e^{t}.1 - e^{-t}.1$

$= 2sin\ ht$

Hence $f(t) = \frac{2\sin ht}{t}$.

6. Find $L^{-1}\left[\frac{s}{s^2+k^2}\right]$

Solution:

$$L^{-1}\left[\frac{s}{s^2+k^2}\right]=\frac{d}{dt}L^{-1}\left[\frac{1}{s^2+k^2}\right]$$

$$=\frac{d}{dt}\left[\frac{\sin kt}{k}\right]=\cos kt$$

Here, $\frac{\sin kt}{k}=0$ when $t = 0$.

7. Find $L^{-1}\left[\frac{s}{(s+3)^2+4}\right]$.

Solution:

$$L^{-1}\left[\frac{s}{(s+3)^2+4}\right]=\frac{d}{dt}L^{-1}\left[\frac{1}{(s+3)^2+4}\right] \quad \text{, Since } L^{-1}[sF(s)]=\frac{d}{dt}L^{-1}[F(s)]$$

$$=\frac{d}{dt}e^{-3t}L^{-1}\left[\frac{1}{s^2+2^2}\right]$$

$$=\frac{d}{dt}e^{-3t}\frac{1}{2}L^{-1}\left[\frac{2}{s^2+2^2}\right]$$

$$=\frac{d}{dt}\frac{1}{2}e^{-3t}\sin 2t$$

$$=\frac{1}{2}(e^{-3t}.2\cos 2t+\sin 2t(-3e^{-3t}))$$

$$=\frac{e^{-3t}}{2}(2\cos 2t-3\sin 2t).$$

8. Find $L^{-1}\left[\frac{s-3}{s^2+4s+13}\right]$.

Solution:

$$L^{-1}\left[\frac{s-3}{s^2+4s+13}\right]=L^{-1}\left[\frac{s}{s^2+4s+13}\right]-L^{-1}\left[\frac{3}{s^2+4s+13}\right]$$

$$=\frac{d}{dt}L^{-1}\left[\frac{1}{s^2+4s+13}\right]-3L^{-1}\left[\frac{1}{s^2+4s+13}\right]$$

$$=\frac{d}{dt}L^{-1}\left[\frac{1}{s^2+4s+4+9}\right]-3L^{-1}\left[\frac{3}{s^2+4s+4+9}\right]$$

$$= \frac{d}{dt} L^{-1}\left[\frac{1}{(s+2)^2+3^2}\right] - 3L^{-1}\left[\frac{3}{(s+2)^2+3^2}\right]$$

$$= \frac{d}{dt}\left[e^{-2t}\frac{\sin 3t}{t}\right] - 3\left[e^{-2t}\frac{\sin 3t}{t}\right]$$

$$= \frac{1}{3}[e^{-2t}\, 3\cos 3t - 2e^{-2t}\sin 3t - 3e^{-2t}\sin 3t]$$

$$= \frac{e^{-2t}}{3}[3\cos 3t - 5\sin 3t]$$

9. Find $L^{-1}\left[\frac{s}{(s+2)^2}\right]$.

Solution:

$$L^{-1}\left[\frac{s}{(s+2)^2}\right] = \frac{d}{dt} L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$= \frac{d}{dt} e^{-2t} L^{-1}\left[\frac{1}{s^2}\right]$$

$$= \frac{d}{dt}(e^{-2t}t)$$

$$= (-2e^{-2t}t + e^{-2t})$$

$$= e^{-2t}(1-2t)$$

10. Find $L^{-1}\left[\frac{s^2}{(s-1)^3}\right]$.

Solution:

$$L^{-1}\left[\frac{s^2}{(s-1)^3}\right] = \frac{d}{dt} L^{-1}\left[\frac{s}{(s-1)^3}\right]$$

$$= \frac{d}{dt}\left(\frac{d}{dt}\left(L^{-1}\left[\frac{1}{(s-1)^3}\right]\right)\right)$$

$$= \frac{d}{dt}(e^t \frac{t^2}{2})$$

$$= \frac{e^t}{2}(t^2 + 4t + 2).$$

Property 3: If $L^{-1}\left[\frac{1}{s}F(s)\right] = \int_0^t L^{-1}[F(s)]\,dt$.

Proof:

w.k.t. $L\left[\int_0^t f(x)dx\right] = \frac{1}{s}L[f(t)]$

Let $F(t) = \int_0^t f(x)dx$. Then $F'(t) = f(t)$ and $F(0) = 0$

$$L[F(t)] = sF(t) - F(0) = sF(t) = s\int_0^t f(x)dx$$

$$\int_0^t f(x)dx = \frac{1}{s}L[F(t)]$$

11. Find $L^{-1}\left[\frac{1}{s(s+a)}\right]$.

Solution:

$L^{-1}\left[\frac{1}{s(s+a)}\right] = \int_0^t L^{-1}\left[\frac{1}{(s+a)}\right]dt$

w.k.t. $L^{-1}\left[\frac{1}{s}F(s)\right] = \int_0^t L^{-1}[F(s)]\,dt$.

$L^{-1}\left[\frac{1}{s(s+a)}\right] = \int_0^t e^{-at}L^{-1}\left[\frac{1}{s}\right]dt$

$= \int_0^t e^{-at}(1)dt$

$= \left[\frac{e^{-at}}{-a}\right]_0^t$

$= -\frac{1}{a}\left[e^{-at} - e^0\right]$

$= \frac{1}{a}(1 - e^{-at})$

12. Find $L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right]$.

Solution:

w.k.t. $L^{-1}\left[\frac{1}{s}F(s)\right] = \int_0^t L^{-1}[F(s)]\,dt.$

$L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right] = \int_0^t L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right]dt$

$=\int_0^t \frac{1}{a}L^{-1}\left[\frac{a}{(s^2+a^2)}\right]dt$

$= \int_0^t \frac{\sin at}{a}dt$

$= \left[-\frac{1}{a}\frac{\cos at}{a}\right]_0^t = -\frac{1}{a^2}(cos\ at - \cos 0)$

$= \frac{1}{a^2}(1-\cos at)$

13. Find $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$.

Solution:

w.k.t. $L^{-1}\left[\frac{1}{s}F(s)\right] = \int_0^t L^{-1}[F(s)]\,dt.$

$L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{1}{s}\frac{s}{(s^2+a^2)^2}\right]$

$= \int_0^t L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]dt$

$= \int_0^t \frac{tsin\ at}{2a}dt$

$= \frac{1}{2a}\left[\frac{-tcos\ at}{a} + \frac{\sin at}{a^2}\right]_0^t$

$= \frac{1}{2a^3}(sinat - atcosat)$

14. Find $L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right]$.

Solution:

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$1 = A(s+1)(s+2) + B\,s(s+2) + c\,s(s+1)$$

Put $s = -1$ we get $B = -1$

Put $s = 0$ we get $A = \frac{1}{2}$

Put $s = -2$ we get $C = \frac{1}{2}$

$$\frac{1}{s(s+1)(s+2)} = \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}$$

$$L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right] = \frac{1}{2}L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{s+2}\right]$$

$$= \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$$

15. Find $L^{-1}\left[\frac{1}{(s+1)(s^2+2s+2)}\right]$.

Solution:

$$\frac{1}{(s+1)(s^2+2s+2)} = \frac{A}{(s+1)} + \frac{Bs+C}{(s^2+2s+2)}$$

$$1 = A(s^2+2s+2) + (Bs+C)(s+1) = A(s^2+2s+2) + B\,s(s+1) + C(s+1)$$

Put $s = -1$ we get $A = 1$

Equating the coefficient of s^2 on both sides we get

$$0 = A + B \Rightarrow B = -1$$

Put $s = 0$ we get $1 = 2A + C \Rightarrow C = -1$

$$\frac{1}{(s+1)(s^2+2s+2)} = \frac{1}{(s+1)} + \frac{-s-1}{(s^2+2s+2)}$$

$$\frac{1}{(s+1)(s^2+2s+2)} = \frac{1}{(s+1)} - \frac{s}{(s^2+2s+2)} - \frac{1}{(s^2+2s+2)}$$

$$L^{-1}\left[\frac{1}{(s+1)(s^2+2s+2)}\right] = L^{-1}\left[\frac{1}{(s+1)}\right] - L^{-1}\left[\frac{s+1}{(s^2+2s+2)}\right]$$

$$= L^{-1}\left[\frac{1}{(s+1)}\right] - L^{-1}\left[\frac{s+1}{(s+1)^2+1^2}\right]$$

$$= e^{-t} - e^{-t}L^{-1}\left[\frac{s}{s^2+1^2}\right]$$

$$= e^{-t} - e^{-t}\cos t$$

$$= e^{-t}(1 - cost)$$

16. Find $L^{-1}\left[\frac{1+2s}{(s+2)^2(s-1)^2}\right]$.

Solution:

$$\frac{1+2s}{(s+2)^2(s-1)^2} = \frac{1}{3}\frac{3(1+2s)}{(s+2)^2(s-1)^2} = \frac{1}{3}\frac{3+6s}{(s+2)^2(s-1)^2}$$

$$= \frac{1}{3}\frac{s^2-s^2+4-1+4s+2s}{(s+2)^2(s-1)^2}$$

$$= \frac{1}{3}\frac{s^2+4s+4-s^2+2s-1}{(s+2)^2(s-1)^2}$$

$$= \frac{1}{3}\frac{s^2+4s+4-(s^2-2s+1)}{(s+2)^2(s-1)^2}$$

$$= \frac{1}{3}\frac{(s+2)^2-(s-1)^2}{(s+2)^2(s-1)^2}$$

$$= \frac{1}{3}\frac{1}{(s-1)^2} - \frac{1}{3}\frac{1}{(s+2)^2}$$

$$L^{-1}\left[\frac{1+2s}{(s+2)^2(s-1)^2}\right] = \frac{1}{3}L^{-1}\left[\frac{1}{(s-1)^2}\right] - \frac{1}{3}L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$= \frac{1}{3}(te^t - te^{-2t})$$

$$= \frac{t}{3}(e^t - e^{-2t}).$$

17. Find $L^{-1}\left[\frac{cs+d}{(s+a)^2+b^2}\right]$.

Solution:

$$L^{-1}\left[\frac{cs+d}{(s+a)^2+b^2}\right] = L^{-1}\left[\frac{cs}{(s+a)^2+b^2}\right] + L^{-1}\left[\frac{d}{(s+a)^2+b^2}\right]$$

$$= cL^{-1}\left[\frac{s}{(s+a)^2+b^2}\right] + dL^{-1}\left[\frac{1}{(s+a)^2+b^2}\right]$$

$$= cL^{-1}\left[\frac{s+a-a}{(s+a)^2+b^2}\right] + dL^{-1}\left[\frac{1}{(s+a)^2+b^2}\right]$$

$$= cL^{-1}\left[\frac{s+a}{(s+a)^2+b^2}\right] - cL^{-1}\left[\frac{a}{(s+a)^2+b^2}\right] + dL^{-1}\left[\frac{1}{(s+a)^2+b^2}\right]$$

$$= ce^{-ax}L^{-1}\left[\frac{s}{s^2+b^2}\right] - acL^{-1}\left[\frac{1}{(s+a)^2+b^2}\right] + dL^{-1}\left[\frac{1}{(s+a)^2+b^2}\right]$$

$$= ce^{-ax}L^{-1}\left[\frac{s}{s^2+b^2}\right] - ace^{-ax}L^{-1}\left[\frac{1}{s^2+b^2}\right] + de^{-ax}L^{-1}\left[\frac{1}{s^2+b^2}\right]$$

$$= \frac{bce^{-ax}}{b}L^{-1}\left[\frac{s}{s^2+b^2}\right] - \frac{ace^{-ax}}{b}L^{-1}\left[\frac{b}{s^2+b^2}\right] + \frac{de^{-ax}}{b}L^{-1}\left[\frac{b}{s^2+b^2}\right]$$

$$= \frac{bce^{-ax}}{b}\cos bx - \frac{ace^{-ax}}{b}\sin bx + \frac{de^{-ax}}{b}\sin bx$$

$$= \frac{e^{-ax}}{b}[bc\ cos\ bx - \text{ac}\sin bx + dsin\ bc].$$

18. Find $L^{-1}\left[\frac{1}{s(s^2-2s+5)}\right]$.

Solution:

w.k.t. $L^{-1}\left[\frac{1}{s}F(s)\right] = \int_0^t L^{-1}[F(s)]\,dt$

$$L^{-1}\left[\frac{1}{s}\frac{1}{s^2-2s+5}\right] = \int_0^t L^{-1}\left[\frac{1}{s^2-2s+5}\right]dt$$

$$= \int_0^t L^{-1}\left[\frac{1}{s^2-2s+1+4}\right]dt$$

$$= \int_0^t L^{-1}\left[\frac{1}{(s-1)^2+2^2}\right]dt.$$

$$= \int_0^t e^t L^{-1}\left[\frac{1}{s^2+2^2}\right] dt$$

$$= \int_0^t e^t \frac{\sin 2t}{2} dt$$

$$= \frac{1}{2}\left[\frac{e^t}{1^2+2^2}(-2\cos 2t + \sin 2t)\right]_0^t$$

$$= \frac{e^t}{10}[\sin 2t - 2\cos 2t] - \frac{e^0}{10}[\sin 0 - 2\cos 0]$$

$$= \frac{e^t}{10}[\sin 2t - 2\cos 2t] - \frac{1}{10}(-2)$$

$$= \frac{e^t}{10}[\sin 2t - 2\cos 2t] + \frac{2}{10}.$$

19. Find $L^{-1}\left[\frac{s^2-s+2}{s(s-3)(s+2)}\right]$.

Solution:

$$\frac{s^2-s+2}{s(s-3)(s+2)} = \frac{A}{s} + \frac{B}{s-3} + \frac{C}{s+2}$$

$$s^2 - s + 2 = A(s-3)(s+2 + B\, s(s+2) + Cs(s-3)$$

Put $s = 0$ we get $A = -\frac{1}{3}$

Put $s = 3$ we get $B = \frac{8}{15}$

Put $s = -2$ we get $C = \frac{4}{5}$

$$L^{-1}\left[\frac{s^2-s+2}{s(s-3)(s+2)}\right] = -\frac{1}{3}L^{-1}\left[\frac{1}{s}\right] + \frac{8}{15}L^{-1}\left[\frac{1}{s-3}\right] + \frac{4}{5}L^{-1}\left[\frac{1}{s+2}\right]$$

$$= -\frac{1}{3}L^{-1}\left[\frac{1}{s}\right] + \frac{8}{15}e^{3t}L^{-1}\left[\frac{1}{s}\right] + \frac{4}{5}e^{-2t}L^{-1}\left[\frac{1}{s}\right]$$

$$= -\frac{1}{3}(1) + \frac{8}{15}e^{3t}(1) + \frac{4}{5}e^{-2t}(1)$$

$$= -\frac{1}{3} + \frac{8}{15}e^{3t} + \frac{4}{5}e^{-2t}.$$

20. Find $L^{-1}\left[\frac{1}{(s-1)(s+3)(s^2+1)}\right]$.

Solution:

$$\frac{1}{(s-1)(s+3)(s^2+1)} = \frac{A}{s-1} + \frac{B}{s+3} + \frac{Cs+D}{s^2+1}$$

$$1 = A(s+3)(s^2+1) + B(s-1)(s^2+1) + (Cs+D)(s-1)(s+3)$$

Put $s = -3$ we get $B = -\frac{1}{40}$

Put $s = 1$ we get $A = \frac{1}{8}$

Put $s = 0$ we get $D = -\frac{1}{5}$

Put $s = -1$ and solving we get $C = -\frac{1}{10}$

$$\frac{1}{(s-1)(s+3)(s^2+1)} = \frac{1}{8}\frac{1}{s-1} - \frac{1}{40}\frac{1}{s+3} - \frac{1}{10}\frac{s}{s^2+1} - \frac{1}{5}\frac{1}{s^2+1}$$

$$L^{-1}\left[\frac{1}{(s-1)(s+3)(s^2+1)}\right] = \frac{1}{8}L^{-1}\left[\frac{1}{s-1}\right] - \frac{1}{40}L^{-1}\left[\frac{1}{s+3}\right] - \frac{1}{10}L^{-1}\left[\frac{s}{s^2+1}\right] - \frac{1}{5}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= \frac{1}{8}e^{x} - \frac{1}{40}e^{-3x} - \frac{1}{10}\cos x - \frac{1}{5}\sin x$$

21. Find $L^{-1}\left[\frac{s+a}{s+b}\right]$.

Solution:

$$L^{-1}\left[\frac{s+a}{s+b}\right] = L^{-1}\left[\frac{s}{s+b}\right] + L^{-1}\left[\frac{a}{s+b}\right]$$

$$= \frac{d}{dt}L^{-1}\left[\frac{1}{s+b}\right] + aL^{-1}\left[\frac{1}{s+b}\right]$$

$$= \frac{d}{dt}(e^{-bt}) + \mathrm{a}e^{-bt}$$

$$= (-\mathrm{b}e^{-bt}) + \mathrm{a}e^{-bt}.$$

TRY YOURSELF

1. Find $L^{-1}\left[\frac{s+3}{(s^2+6s+13)^2}\right]$.

SOLVING DIFFERENTIAL EQUATIONS USING LAPLACE TRANSFORM

1. **Using Laplace transform solve $\frac{d^2y}{dt^2}+2\frac{dy}{dt}-3y=sint$ given that $y=\frac{dy}{dt}=0$.**

Solution:

Given $y''+2y'-3y=sint$

Taking Laplace transform on both sides

$L[y''+2y'-3y]=L[sint]$

$L[y'']+2L[y']-3L[y]=L[sint]$

$(s^2L[y]-sy(0)-y'(0))+2\big(sL[y]-y(0)\big)-3L[y]=\frac{1}{s^2+1}$

$(s^2L[y]-s(0)-0)+2(sL[y]-0)-3L[y]=\frac{1}{s^2+1}$

$s^2L[y]+2sL[y]-3L[y]=\frac{1}{s^2+1}$

$L[y](s^2+2s-3)=\frac{1}{s^2+1}$ ➔ $L[y](s+3)(s-1)=\frac{1}{s^2+1}$

$L[y]=\frac{1}{(s+3)(s-1)(s^2+1)}$

$\frac{1}{(s+3)(s-1)(s^2+1)}=\frac{A}{(s+3)}+\frac{B}{(s-1)}+\frac{Cs+D}{(s^2+1)}$

$1=A(s-1)(s^2+1)+B(s+3)(s^2+1)+(Cs+D)(s+3)(s-1)$

Put $s=1$ we get $B=\frac{1}{8}$

Put $s=-3$ we get $A=-\frac{1}{40}$

Put $s=0$ we get $D=-\frac{1}{5}$

Equating the coefficient of s^3 on both sides we get

$A + B + C = 0$. Solving we get $C = -\frac{1}{10}$

$$L[y] = -\frac{1}{40}\frac{1}{(s+3)} + \frac{1}{8}\frac{1}{(s-1)} - \frac{1}{10}\frac{s}{(s^2+1)} - \frac{1}{5}\frac{1}{(s^2+1)}$$

$$y = -\frac{1}{40}L^{-1}\left[\frac{1}{(s+3)}\right] + \frac{1}{8}L^{-1}\left[\frac{1}{(s-1)}\right] - \frac{1}{10}L^{-1}\left[\frac{s}{(s^2+1)}\right] - \frac{1}{5}L^{-1}\left[\frac{1}{(s^2+1)}\right]$$

$$y = -\frac{1}{40}e^{-3t} + \frac{1}{8}e^{t} - \frac{1}{10}\cos t - \frac{1}{5}\sin t$$

2. Using Laplace transform solve $\frac{d^2y}{dt^2} + 4y = A \sin k\, t$ given that $y(0) = y'(0) = 0$ when $t = 0$.

Solution:

Given $y'' + 4y = A \sin k\, t$

Taking Laplace transform on both sides

$$L[y'' + 4y] = L[A \sin k\, t]$$

$$L[y''] + 4L[y] = AL[\sin k\, t]$$

$$(s^2 L[y] - sy(0) - y'(0)) + 4L[y] = A\frac{k}{s^2+k^2}$$

$$(s^2 L[y] - s(0) - 0) + 4L[y] = \frac{Ak}{s^2+k^2}$$

$$s^2 L[y] + 4L[y] = \frac{Ak}{s^2+k^2}$$

$$(s^2 + 4)L[y] = \frac{Ak}{s^2+k^2} \Rightarrow L[y] = \frac{Ak}{(s^2+k^2)(s^2+4)}$$

$$y = L^{-1}\left[\frac{Ak}{(s^2+k^2)(s^2+4)}\right] = AkL^{-1}\left[\frac{1}{(s^2+k^2)(s^2+4)}\right]$$

Case (i): $k \neq 2$

$$\frac{1}{(s^2+k^2)(s^2+4)} = \frac{As+B}{(s^2+4)} + \frac{Cs+D}{(s^2+k^2)}$$

$1 = As(s^2+k^2)+B(s^2+k^2)+Cs(s^2+4)+D(s^2+4)$

Equating the coefficient of s^3 on both sides we get $0 = A + C$

Equating the coefficient of s^2 on both sides we get $0 = B + D$

Equating the coefficient of s on both sides we get $0 = Ak^2 + 4C$

Solving these equations we get $A = 0, C = 0$

Put $s = 0$ we get $Bk^2 + 4D = 0$

Since $B = -D$ we get $B = \frac{1}{k^2-4}$ and D = $-\frac{1}{k^2-4}$

$$y = AkL^{-1}\left[\frac{1}{(s^2+k^2)(s^2+4)}\right] = AkL^{-1}\left[\frac{\frac{1}{k^2-4}}{(s^2+4)}\right] - AkL^{-1}\left[\frac{\frac{1}{k^2-4}}{(s^2+k^2)}\right]$$

$$y = \frac{Ak}{k^2-4}\left(L^{-1}\left[\frac{1}{(s^2+2^2)}\right] - L^{-1}\left[\frac{1}{(s^2+k^2)}\right]\right)$$

$$y = \frac{Ak}{k^2-4}\left(\frac{\sin 2t}{2} - \frac{\sin kt}{k}\right)$$

Case (ii): $k = 2$

$$y = A(2)L^{-1}\left[\frac{1}{(s^2+2^2)(s^2+4)}\right]$$

$$=2AL^{-1}\left[\frac{1}{(s^2+2^2)^2}\right] = 2AL^{-1}\left[\frac{s}{s(s^2+2^2)^2}\right]$$

$$y = 2A\int_0^t L^{-1}\left[\frac{s}{s(s^2+2^2)^2}\right]\, dt$$

w.k.t. $L^{-1}\left[\frac{1}{s}F(s)\right] = \int_0^t L^{-1}[F(s)]\, dt.$

$$y = \frac{2A}{4}\int_0^t L^{-1}\left[\frac{4s}{(s^2+2^2)^2}\right]\, dt$$

$$y = \frac{2A}{4}\int_0^t t\sin 2t\;\, dt$$

$y = \frac{2A}{4}[uv - u'v_1 + u''v_2 - \cdots]$ where $u = t, dv = \sin 2t dt$

$y = \frac{2A}{4}\left[-\frac{t\cos 2t}{2} + \frac{\sin 2t}{4}\right]_0^t$

$y = \frac{2A}{4}\left[\frac{-2t\cos 2t + \sin 2t}{4}\right]_0^t$

$y = \frac{2A}{16}(\sin 2t - 2\cos 2t) = \frac{A}{8}(\sin 2t - 2\cos 2t)$

3. Using Laplace transform solve $3\frac{dx}{dt} + \frac{dy}{dt} + 2x = 1, \frac{dx}{dt} + 4\frac{dy}{dt} + 3y = 0$ given that $x = y = 0$ when $t = 0$.

Solution:

Given $3x' + y' + 2x = 1, x' + 4y' + 3y = 0$

Taking Laplace transform on both sides

$L[3x' + y' + 2x] = L[1]$

$3L[x'] + L[y'] + 2L[x] = L[1]$

3(s $L[x] - x(0)$)+(s $L[y] - y(0)$)+2 $L[x] = L[1]$

3s $L[x] + sL[y] + 2L[x] = \frac{1}{s}$

$(3s + 2)L[x] + sL[y] = \frac{1}{s}$ ------------------------(1)

Also $x' + 4y' + 3y = 0$

$L[x' + 4y' + 3y] = L[0]$

$L[x'] + 4L[y'] + 3L[y] = 0$

(s $L[x] - x(0)$)+4($sL[y] - y(0)$)+3 $L[y]$=0

$sL[x] + 4sL[y] + 3L[y] = 0$

$sL[x] + (4s + 3)L[y] = 0$

$(4s + 3)L[y] = -sL[x]$

$L[x] = -\frac{(4s+3)L[y]}{s}$

From (1), $(3s+2)\left(-\frac{(4s+3)L[y]}{s}\right) + sL[y] = \frac{1}{s}$

$\frac{-(3s+2)(4s+3)L[y]+s^2L[y]}{s} = \frac{1}{s}$

$\frac{(-12s^2-9s-8s-6)L[y]+s^2L[y]}{s} = \frac{1}{s}$

$\frac{(-11s^2-17s-6)L[y]}{s} = \frac{1}{s}$ ➔ $-\frac{(11s^2+17s+6)L[y]}{s} = \frac{1}{s}$

$L[y] = \frac{1}{s} \times \frac{-s}{(11s^2+17s+6)} = \frac{-1}{(s+1)(11s+6)}$

$L[x] = -\frac{(4s+3)}{s} L[y]$

$= -\frac{(4s+3)}{s} \frac{-1}{(s+1)(11s+6)}$

$= \frac{4s+3}{s(s+1)(11s+6)}$

$\frac{4s+3}{s(s+1)(11s+6)} = \frac{A}{s} + \frac{B}{(s+1)} + \frac{C}{(11s+6)}$

$4s+3 = A(s+1)(11s+6) + Bs(11s+6) + Cs(s+1)$

Put $s = 0$ we get $A = \frac{1}{2}$

Put $s = -1$ we get $B = -\frac{1}{5}$

Put $s = -\frac{6}{11}$ we get $C = -\frac{33}{10}$

$L[x] = \frac{1}{2}\frac{1}{s} - \frac{1}{5}\frac{1}{(s+1)} - \frac{33}{10}\frac{1}{(11s+6)}$

$x = \frac{1}{2}L^{-1}\left[\frac{1}{s}\right] - \frac{1}{5}L^{-1}\left[\frac{1}{(s+1)}\right] - \frac{33}{10}L^{-1}\left[\frac{1}{(11s+6)}\right]$

$= \frac{1}{2}L^{-1}\left[\frac{1}{s}\right] - \frac{1}{5}L^{-1}\left[\frac{1}{(s+1)}\right] - \frac{33}{10(11)}L^{-1}\left[\frac{1}{\left(s+\frac{6}{11}\right)}\right]$

$x = \frac{1}{2}(1) - \frac{1}{5}(e^{-t}) - \frac{3}{10}e^{-6/11t}$

$L[y] = -\frac{1}{(s+1)(11s+6)} = \frac{A}{(s+1)} + \frac{B}{(11s+6)}$

$1 = A(11s+6) + B(s+1)$

Put $s = -\frac{6}{11}$ we get B=$\frac{11}{5}$

Put $s = -1$ we get $A = -\frac{1}{5}$

$y = -L^{-1}\left[\frac{A}{(s+1)} + \frac{B}{(11s+6)}\right]$

$= -L^{-1}\left[-\frac{1}{5}\frac{1}{(s+1)} + \frac{11}{5}\frac{1}{(11s+6)}\right]$

$= \frac{1}{5}L^{-1}\left[\frac{1}{(s+1)}\right] - \frac{11}{5(11)}L^{-1}\left[\frac{1}{\left(s+\frac{6}{11}\right)}\right]$

$= \frac{1}{5}L^{-1}\left[\frac{1}{(s+1)}\right] - \frac{1}{5}L^{-1}\left[\frac{1}{\left(s+\frac{6}{11}\right)}\right]$

$y = \frac{1}{5}e^{-t} - \frac{1}{5}e^{-6/11t}$

4. Using Laplace transform solve $\frac{dx}{dt} - \frac{dy}{dt} - 2x + 2y = 1 - 2t, \frac{d^2x}{dt^2} + 2\frac{dy}{dt} + x = 0$ given that $x = y = x' = 0$ when $t = 0$.

Solution:

Given $x' - y' - 2x + 2y = 1 - 2t$

Taking Laplace transform on both sides

$L[x' - y' - 2x + 2y] = L[1 - 2t]$

$L[x'] - L[y'] - 2L[x] + 2L[y] = L[1] - 2L[t]$

(s $L[x] - x(0)$) $-$(s $L[y] - y(0)$) $-2\,L[x] + 2L[y] = L[1] - 2L[t]$

$s\,L[x] - s\,L[y] - 2L[x] + 2L[y] = \frac{1}{s} - 2\left(\frac{1}{s^2}\right)$

$s\,L[x] - 2L[x] - s\,L[y] + 2L[y] = \frac{s}{s^2} - \left(\frac{2}{s^2}\right)$

$L[x](s-2) - L[y](s-2) = \frac{(s-2)}{s^2}$

$L[x] - L[y] = \frac{1}{s^2}$ ➔ $L[x] = \frac{1}{s^2} + L[y]$ ----------------------(1)

Given $x'' + 2y' + x = 0$

Taking Laplace transform on both sides we get

$L[x'' + 2y' + x] = L[0]$

$L[x''] + 2L[y'] + L[x] = L[0]$

$(s^2L[x] - sx(0) - x'(0)) + 2\,(s\,L[y] - y(0)) + L[x] = 0$

$(s^2L[x] - 0 - 0) + 2\,(s\,L[y] - 0) + L[x] = 0$

$s^2L[x] + 2s\,L[y] + L[x] = 0$

$(s^2 + 1)\,L[x] = -2sL[y]$

Substituting value of $L[x]$ from (1) we get

$(s^2 + 1)\left(\frac{1}{s^2} + L[y]\right) = -2sL[y]$

$1 + s^2L[y] + \frac{1}{s^2} + L[y] + 2sL[y] = 0$

$1 + \frac{1}{s^2} + s^2L[y] + L[y] + 2sL[y] = 0$

$\frac{s^2+1}{s^2} + (s^2 + 1)L[y] + 2sL[y] = 0$

$(s^2 + 2s + 1)L[y] = -\frac{s^2+1}{s^2}$

$L[y] = -\frac{s^2+1}{s^2(s^2+2s+1)}$

From (1), $L[x] = \frac{1}{s^2} + L[y]$

$$= \frac{1}{s^2} - \frac{s^2+1}{s^2(s^2+2s+1)}$$

$$= \frac{s^2+2s+1-s^2-1}{s^2(s^2+2s+1)}$$

$$L[x] = \frac{2}{s(s+1)^2} \Rightarrow x = L^{-1}\left[\frac{1}{s}\frac{2}{(s+1)^2}\right]$$

w.k.t. $L^{-1}\left[\frac{1}{s}F(s)\right] = \int_0^t L^{-1}[F(s)]\,dt.$

$$x = \int_0^t L^{-1}\left[\frac{2}{(s+1)^2}\right] dt$$

$$= 2\int_0^t e^{-t}L^{-1}\left[\frac{1}{s^2}\right] dt$$

$$= 2\int_0^t e^{-t}\,tdt$$

$$= 2[-te^{-t} - 1.e^{-t}]_0^t$$

$$= 2[(-te^{-t} - e^{-t}) - (0 - e^0)]$$

$$= 2[(-te^{-t} - e^{-t}) + 1]$$

$$= 2[(1 - e^{-t} - te^{-t})]$$

$$y = -L^{-1}\left[\frac{s^2+1}{s^2(s^2+2s+1)}\right]$$

$$= -L^{-1}\left[\frac{1}{s}\frac{s^2+1}{s(s+1)^2}\right]$$

$$= -\int_0^t L^{-1}\left[\frac{s^2+1}{s(s+1)^2}\right] dt$$

$$\frac{s^2+1}{s(s+1)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

$$s^2 + 1 = A(s+1)^2 + Bs(s+1) + Cs$$

Put $s = 0$ we get $A = 1$

Put $s = -1$ we get $C = -2$

Equating the coefficient of s^2 on both sides

$A + B = 1 \rightarrow B = 0$

$\frac{s^2+1}{s(s+1)^2} = \frac{1}{s} - \frac{2}{(s+1)^2}$

$y = -\int_0^t L^{-1}\left[\frac{s^2+1}{s(s+1)^2}\right] dt$

$= -\int_0^t L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{2}{(s+1)^2}\right] dt$

$= -\int_0^t 1 - 2e^{-t}L^{-1}\left[\frac{1}{s^2}\right] dt$

$= -\int_0^t (1 - 2e^{-t}t)\, dt$

$= -\int_0^t dt + 2\int_0^t te^{-t}\, dt$

$= [-t + 2(te^{-t} - e^{-t})]_0^t$

$= -t + 2(te^{-t} - e^{-t}) - \left(0 + 2(0 - e^0)\right)$

$= -t + 2(te^{-t} - e^{-t}) - (2(-1))$

$y = -t + 2(te^{-t} - e^{-t}) + 2$

5. Using Laplace transform solve $t\frac{d^2y}{dt^2} - (2+t)\frac{dy}{dt} + 3y = t - 1$.

Solution:

Given $ty'' - (2+t)y' + 3y = t - 1$

Taking Laplace transform on both sides we get

$L[ty'' - (2+t)y' + 3y] = L[t - 1]$

$L[ty''] - (2+t)L[y'] + 3L[y] = L[t - 1]$

$L[ty''] - 2L[y'] - L[ty'] + 3L[y] = L[t-1]$

$-\frac{d}{ds}L[y''] - 2L[y'] + \frac{d}{ds}L[y'] + 3L[y] = L[t] - L[1]$

$-\frac{d}{ds}(s^2L[y] - sy(0) - y'(0)) - 2\big(s\,L[y] - y(0)\big) +$

$$\frac{d}{ds}(s\,L[y] - y(0)) + 3L[y] = \frac{1}{s^2} - \frac{1}{s}$$

$-\frac{d}{ds}(s^2L[y]) - 2(s\,L[y]) - 2sL[y] + \frac{d}{ds}(s\,L[y]) + 3L[y] = \frac{1-s}{s^2}$

$-s^2\frac{d}{ds}L[y] - L[y].2s) - 2sL[y] + (s\frac{d}{ds}L[y] + L[y].1) + 3L[y] = \frac{1-s}{s^2}$

$-s^2\frac{d}{ds}L[y] + s\,\frac{d}{ds}L[y] - 2sL[y] - 2sL[y] + L[y] + 3L[y] = \frac{1-s}{s^2}$

$(s - s^2)\frac{d}{ds}L[y] - 4sL[y] + 4L[y] = \frac{1-s}{s^2}$

$-s(s-1)\frac{d}{ds}L[y] - 4(s-1)L[y] = -\frac{s-1}{s^2}$

$-s\frac{d}{ds}L[y] - 4L[y] = -\frac{1}{s^2}$

$\frac{d}{ds}L[y] + \frac{4}{s}L[y] = \frac{1}{s^3}$

This is linear D.E in L[y] with $P = \frac{4}{s}$, $Q = \frac{1}{s^3}$

Integrating factor = $e^{\int Pdx} = e^{\int\frac{4}{s}ds}$

I. F = $e^{4logs} = e^{logs^4} = s^4$

$L[y]$. (I.F) = $\int (I.F)Qds + C$

$s^4\,L[y] = \int s^4\frac{1}{s^3}ds + C$

$s^4\,L[y] = \int sds + C = \frac{s^2}{2} + C$

$L[y] = \frac{s^2}{2s^4} + \frac{C}{s^4} = \frac{1}{2s^2} + \frac{C}{s^4}$

$y = \frac{1}{2} L^{-1}\left[\frac{1}{s^2}\right] + CL^{-1}\left[\frac{1}{s^4}\right]$

$= \frac{1}{2}.t + C\frac{t^3}{3!}$

$= \frac{t}{2} + \frac{ct^3}{3!}.$

6. Using Laplace transform solve $\frac{d^2y}{dt^2} + t\frac{dy}{dt} - y = 0$ if $y(0) = 0, y'(0) = 1$.

Solution:

Given $y'' + ty' - y = 0$

Taking Laplace transform on both sides, we get

$L[y'' + ty' - y] = L[0]$

$L[y''] + L[ty'] - L[y] = 0$

$(s^2L[y] - sy(0) - y'(0)) - \frac{d}{ds}L[y'] - L[y] = 0$

$(s^2L[y] - sy(0) - y'(0)) - \frac{d}{ds}\big(s\,L[y] - y(0)\big) - L[y] = 0$

$(s^2L[y] - 1) - \frac{d}{ds}(s\,L[y]) - L[y] = 0$

$(s^2L[y] - 1) - \left(s\frac{d}{ds}\,L[y] + L[y].1\right) - L[y] = 0$

$(s^2L[y] - 1) - s\frac{d}{ds}\,L[y] - L[y].1 - L[y] = 0$

$s^2L[y] - 1 - s\frac{d}{ds}\,L[y] - 2L[y] = 0$

$-s^2L[y] + 1 + s\frac{d}{ds}\,L[y] + 2L[y] = 0$

$s\frac{d}{ds}\,L[y] - (s^2 - 2)L[y] = 0$

$\frac{d}{ds}\,L[y] - \frac{s^2-2}{s}L[y] = 0$

Which is linear in $L[y]$ where P = $-\frac{s^2-2}{s} = \frac{2}{s} - s$ Q = $-\frac{1}{s}$

Integrating factor = $e^{\int Pdx} = e^{\int(\frac{2}{s}-s)ds}$

I. F = $e^{2logs-\frac{s^2}{2}} = e^{logs^2-\frac{s^2}{2}} = e^{logs^2}.e^{-\frac{s^2}{2}} = s^2.e^{-\frac{s^2}{2}}$

$L[y]$. (I.F) = $\int(I.F)Qds + C$

$$L[y]\left(s^2.e^{-\frac{s^2}{2}}\right) = \int\left(s^2.e^{-\frac{s^2}{2}}\right)(-\frac{1}{s})ds$$

$$= -\int\left(s.e^{-\frac{s^2}{2}}\right)ds$$

Put $t = -\frac{s^2}{2}$ ➔ $dt = -\frac{2sds}{2} = -sds$

$$L[y]\left(s^2.e^{-\frac{s^2}{2}}\right) = -\int se^t\frac{dt}{-s} = \int e^t dt$$

$$=e^t + C$$

$$= e^{-\frac{s^2}{2}} + C$$

$$L[y] = \frac{e^{-\frac{s^2}{2}}}{s^2.e^{-\frac{s^2}{2}}} + \frac{C}{s^2.e^{-\frac{s^2}{2}}}$$

$$= \frac{1}{s^2} + \frac{Ce^{\frac{s^2}{2}}}{s^2}$$

As $s \to \infty$, $L[y] \to 0$. Hence C = 0

$$L[y] = \frac{1}{s^2}$$

Hence $y = L^{-1}\left[\frac{1}{s^2}\right] = t.$

7. **Solve** $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} - 5y = 5$ **if** $y(0) = 0, y'(0) = 2$.

Solution:

Given $y'' + 4y' - 5y = 5$

$L[y'' + 4y' - 5y] = L[5]$

$L[y''] + 4L[y'] - 5L[y] = L[5]$

$(s^2L[y] - sy(0) - y'(0)) + 4(s\,L[y] - y(0)) - 5L[y] = L[5]$

$s^2L[y] - 2 + 4s\,L[y] - 5L[y] = 5L[1]$

$(s^2 - 4s - 5)L[y] - 2 = 5\frac{1}{s}$

$(s+5)(s-1)L[y] = \frac{5}{s} + 2$

$L[y] = \frac{5}{s(s+5)(s-1)} + \frac{2}{(s+5)(s-1)}$

$\frac{5}{s(s+5)(s-1)} = \frac{A}{s} + \frac{B}{s+5} + \frac{C}{s-1}$

Solving we get $A = -1$ $B = \frac{1}{6}, C = \frac{5}{6}$

$\frac{2}{(s+5)(s-1)} = \frac{D}{(s+5)} + \frac{E}{(s-1)}$

Solving we get $D = \frac{1}{3}, E = -\frac{1}{2}$

$L[y] = -\frac{1}{s} + \frac{1}{6}\frac{1}{s+5} + \frac{5}{6}\frac{1}{s-1} + \frac{1}{3}\frac{1}{s+5} - \frac{1}{2}\frac{1}{s-1}$

$L[y] = -\frac{1}{s} + \frac{1}{6}\frac{1}{s+5} + \frac{2}{6}\frac{1}{s+5} + \frac{5}{6}\frac{1}{s-1} - \frac{3}{6}\frac{1}{s-1}$

$L[y] = -\frac{1}{s} + \frac{1}{2}\frac{1}{s+5} + \frac{1}{3}\frac{1}{s+5}$